Riv. Mat. Univ. Parma (4) 11 (1985), 101-109

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On regular rings and Artinian rings (II) (**)

Introduction

The concept of injectivity is among the most important fundamental concepts of the theory of rings and modules (cf. for example, [2], [3] and [5]), motivating active research on injectivity since several years. In this note, we introduce a generalization of injective modules, noted YJ-injective, to be considered in connection with Δ-rings, Σ-rings and Kasch rings.

Throughout, A represents an associative ring with identity and A-modules are unitary. J, Z, Y will stand respectively for the Jacobson radical, the left singular and the right singular ideal of A. As usual, an ideal of A means a two-sided ideal and A is called *left duo* iff every left ideal of A is an ideal. A left (right) ideal of A is called *reduced* if it contains no non-zero nilpotent element.

We now introduce the following generalization of injective modules.

Def. A right A-module M is called *YJ-injective* if, for any $0 \neq a \in A$, there exists a positive integer n such that $a^n \neq 0$ and any right A-homomorpnism of $a^n A$ into M extends to one of A into M.

Left YJ-injective modules are similarly defined. A direct summand of a right YJ-injective module is YJ-injective. It is easy to see that if A is von Neumann regular, then every right (left) A-module is YJ-injective. We do not know whether the converse is true. However, if A is either reduced or left duo, then the answer is positive (cf. Theorem 5.1). Kasch rings, Δ-rings and

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Σ-rings ([**2**]$_2$) with certain YJ-injective conditions are also considered (Theorem 7). But we start with a necessary and sufficient condition for classical left quotient rings to be strongly regular. Recall that Q is a classical left quotient ring of A iff Q is a ring containing A such that every non-zero-divisor of A is invertible in Q and every element of Q is of the form $q = b^{-1}a$, b, $a \in A$, b being a non-zero-divisor of A.

Proposition 1. *Let A have a classical left quotient ring Q. The following conditions are then equivalent:*

(1) *Q is strongly regular.*

(2) *A is reduced such that for any $a \in A$, there exists an idempotent $e \in Q$ such that $Aa \subseteq l_A(e) \subseteq l_A(r_A(a))$.*

Proof. Assume (1). For any $a \in A$, $a = aqa$, $q \in Q$ and $u = qa$ is idempotent in Q such that $Qa = Qu = l_Q(e)$, where $e = 1 - u$. If $c \in l_A(e)$, $c = cu = cqa$ which yields $l_A(e) \subseteq l_A(r_A(a))$ and therefore (1) implies (2).

Assume (2). Let $q = s^{-1}a \in Q$, s, $a \in A$. By hypothesis, $Aa \subseteq l_A(e) \subseteq l_A(r_A(a))$ for some idempotent $e \in Q$. Set $u = 1 - e$, $b = a + e$. Since A is reduced, then so is Q and we then get $l_A(b) = r_A(b) = 0$. If $b = c^{-1}y$, c, $y \in A$, then $r_A(y) = 0$ which implies that y is invertible in Q and since $ba = a^2$, we have $a = a(y^{-1}c)a$ (Q being reduced), whence $q = q(y^{-1}cs)q$ which proves that (2) implies (1).

If N is a left submodule of a left A-module M, write $K_M(N) = \{y \in M \setminus cy \in N$ for some non-zero-divisor c of $A\}$. If A has a classical left quotient ring, then $K_M(N)$ is a submodule of ${}_AM$. The usual closure of N in M is $Cl_M(N) = \{y \in M \setminus Ly \subseteq N$ for some essential left ideal L of $A\}$. In general, $Cl_M(N) \neq K_M(N)$. It is well-known that A has a classical left quotient ring iff A satisfies the left Ore condition ([**3**]$_1$, p. 101). The next proposition shows that rings whose essential and complement left ideals are ideals must have classical left quotient rings.

Proposition 2. *Suppose that all essential and complement left ideals of A are ideals. The following are then equivalent:*

(1) *A is a reduced left Goldie ring.*

(2) *For any left A-module M and every left submodule N, $Cl_M(N) = K_M(N)$.*

(3) *Every essential left ideal of A contains a non-zero-divisor.*

(4) *For every left ideal I of A, $K_A(I)$ is a complement left ideal.*

Proof. Let a, $c \in A$, c non-zero-divisor. If K is a complement left ideal such that $Ac \oplus K$ is an essential left ideal, then $Kc \subseteq K \cap Ac = 0$ implies

$K = 0$, whence Ac is essential and is therefore an ideal of A. Then $ca = dc$ for some $d \in A$, which implies that A satisfies the left Ore condition.

(1) implies (2) by ([**3**]$_1$, Theorem 3.34).

If L is an essential left ideal, then $Cl_A(L) = A$ and therefore (2) implies (3).

Assume (3). If I is a left ideal of A, let E be an essential extension of $K_A(I)$ in ${}_AA$. For any $y \in E$, $Ly \subseteq K_A(I)$ for some essential left ideal L, whence L contains a non-zero-divisor c and $cy \in K_A(I)$ implies there exists a non-zero-divisor b such that $bcy \in I$. Therefore $y \in K_A(I)$ which proves that (3) implies (4).

Assume (4). If T is an essential left ideal of the classical left quotient ring Q, then $U = T \cap A$ is an essential left ideal of A which therefore contains a non-zero-divisor (because $K_A(U) = A$), whence $QU = Q$. This yields $T = Q$ and (4) implies (1) by ([**8**]$_1$, Lemma 1).

We now turn to YJ-injectivity.

Lemma 3. *The following conditions are equivalent*:

(1) *A is a YJ-injective right A-module.*

(2) *For any $0 \neq a \in A$, there exists a positive integer n such that Aa^n is a non-zero left annihilator.*

Proof. (1) implies (2). For any $0 \neq b \in A$, there exists a positive integer n such that $b^n \neq 0$ and for any $u \in l(r(Ab^n))$, since $r(b^n) = r(l(r(b^n))) \subseteq r(u)$, the right A-homomorphism g of b^nA into A defined by $g(b^n a) = ua$ $(a \in A)$ yields $u = g(b^n) = yb^n$ for some $y \in A$, whence $l(r(Ab^n)) = Ab^n$.

(2) implies (1). If $c \in A$, n a positive integer such that Ac^n is a non-zero left annihilator, let $f: c^nA \to A$ be any right A-homomorphism. Then $r(c^n) \subseteq r(f(c^n))$ implies $Af(c^n) \subseteq l(r(f(c^n))) \subseteq l(r(c^n)) = Ac^n$ and hence $f(c^n) = dc^n$ for some $d \in A$ which proves that A_A is YJ-injective.

A is called *a right YJ-injective ring* iff A_A is YJ-injective. Right YJ-injective rings generalize right self-injective rings and rings whose injective left modules are flat. Following [**2**]$_1$, an element c of A is called *right regular* iff $r(c) = 0$. Then c is a non-zero-divisor iff it is right and left regular. It is well-known that if A is right self-injective, then $Y = J$ ([**2**]$_1$, Corollary 19.28). For right YJ-injective rings, we have

Remark 1. Let A be right YJ-injective. Then (a) $Y = J$; (b) a right regular element of A is left invertible and consequently, any left or right A-module is divisible.

Remark 2. If A is a commutative ring such that every finitely generated ideal is either a maximal annihilator or a projective annihilator, then A is either quasi-Frobeniusean or von Neumann regular.

Applying [**7**] (Proposition 1), Lemma 3 and Remark 1, we get

Proposition 3.1. *The following conditions are equivalent:*

(1) *A is quasi-Frobeniusean.*

(2) *A is a right Noetherian, right YJ-injective ring whose right ideals are right annihilators.*

(3) *A is a right Artinian, left and right YJ-injective ring.*

Following $[\mathbf{2}]_2$, A is called a *left Δ-ring* (resp. *Σ-ring*) iff the set of left ideals of A which are left annihilators of subsets of the injective hull of ${}_AA$ satisfies the descending (resp. ascending) chain condition. Consequently, left Δ (resp. Σ)-rings generalize left Artinian (resp. Noetherian) rings.

Combining ($[\mathbf{2}]_2$, theorems 11.4.1 and 11.4.4), Lemma 3 and Proposition 3.1, we get

Corollary 3.2. *The following conditions are equivalent for a commutative ring A:*

(1) *A is quasi-Frobeniusean.*

(2) *A is a Σ-ring whose principal ideals are annihilators.*

(3) *A is a YJ-injective Δ-ring.*

Remark 3. The following conditions are equivalent for a left duo ring A: (a) A is a semi-prime left Δ-ring. (b) A is a semi-prime left Σ-ring. (c) For any left A-module M and left submodule N, $Cl_M(N) = K_M(N)$. (Apply $[\mathbf{2}]_2$, Theorem 11.4.9 to Proposition 2).

Recall that A is a right uniform ring iff every non-zero right ideal is essential.

Proposition 4. *Let A be a right uniform right YJ-injective ring. Then A is a local ring and $Y = J$ is the unique maximal left (and right) ideal of A.*

Proof. If $Y = 0$, then A is a right Ore domain and by Remark 1(b), A is a division ring. Now suppose that $Y \neq 0$. For any $a \in A$, $a \notin Y$, $r(a) = 0$ which implies a left invertible (Remark 1(b)). Therefore every proper left ideal (in particular, every maximal left ideal) is contained in Y, which proves the proposition.

Corollary 4.1. *A right Noetherian right uniform right YJ-injective ring is right Artinian local.*

Lemma 5. *Suppose that A satisfies any one of the following conditions: (1) A is left YJ-injective or (2) every maximal left ideal of A is YJ-injective. Then any reduced principal left ideal of A is generated by an idempotent.*

(The proof depends on the fact that if Ab is a reduced principal left ideal, then $r(b^n) \subseteq l(b)$ and $l(b^n) = l(b)$ for any positive integer n).

Remark 4. ([1], Corollary 6) holds for the following classes of rings A: (1) Every maximal right ideal of A is YJ-injective. (2) Every non-zero reduced right ideal of A contains a non-zero principal YJ-injective right ideal.

We are now in a position to give some new characteristic properties of strongly regular rings.

Theorem 5.1. *The following conditions are equivalent:*

(1) *A is strongly regular.*

(2) *For any $a \in A$, there exists a central idempotent $e \in A$ satisfying $K_A(Aa) = Aa \subseteq l(e) \subseteq l(r(a))$.*

(3) *For any maximal left ideal M of A and any $a \in M$, there exist a central idempotent $e \in M$ and a left regular element c of A such that $a = ec$.*

(4) *A is a left duo ring such that the sum of any two injective left A-modules is YJ-injective and flat.*

(5) *A is a left duo ring whose simple left modules are YJ-injective.*

(6) *A is a left duo ring whose simple right modules are YJ-injective.*

(7) *A is a reduced ring whose maximal left ideals are YJ-injective.*

(8) *A is a left duo left YJ-injective ring containing a reduced maximal left ideal.*

Proof. It is easily seen that $K_A(Aa) = Aa$ for each $a \in A$ iff every non-zero-divisor is invertible in A. Therefore (1) implies (2) by Proposition 1.

Assume (2). It is sufficient to show that A is reduced for then (2) will imply (1) by Proposition 1. Suppose there exists $a \in A$ such that $a^2 = 0$. Since $Aa \subseteq l(e) \subseteq l(r(a))$ for some central idempotent e, then $l(e) = l(r(a))$, which implies $r(a) = eA$, whence $a = ae = 0$, proving that A is reduced.

Assume (1). Let M be a maximal left ideal of A, $a \in M$. Then $Aa = Av = l(u)$, where v is a central idempotent and $u = 1 - v$, whence $c = a + u$ is a non-zero-divisor and therefore invertible in A. Now $ac = a^2$ implies $a = a^2c^{-1}$, whence $a = ac^{-1}a$, yielding $a = ec$, where $e = c^{-1}a$ is idempotent in M. Thus (1) implies (3).

Assume (3). Let $b \in A$ such that $b^2 = 0$. If $Ab \neq A$, by hypothesis, $b = ec$, where e is a central idempotent and c is a left regular element. Then $0 = ec^2 = bc$ implies $b = 0$, which proves that A is reduced. If M is a maximal left ideal, for any $a \in M$, there exist a central idempotent $u \in M$ and $d \in A$ with $l(d) = 0$ such that $a = ud$. Then $a = udu \in aM$ which implies that ${}_AA/M$ is flat, whence (3) implies (4) by [**8**]$_2$ (Theorem 1.4).

Assume (4). Since the sum of any two injective left A-modules is YJ-injective, then any quotient module of an injective left A-module is YJ-injective (cf. the proof of [**8**]$_6$, theorem 11(6)). If $Z \neq 0$, by [**8**]$_7$ (Lemma 7), there exists $0 \neq z \in Z$ such that $z^2 = 0$. Let E denote an injective left A-module, N a left submodule of E, $f\colon Az \to E/N$ a left A-homomorphism, $k\colon E \to E/N$ the natural projection. Since ${}_AE/N$ is YJ-injective, we get a left A-homomorphism $g\colon Az \to E$ such that $kg = f$. Then, using this property, it can be proved that if M is a left A-module, S a left submodule of M, $F\colon Az \to M/S$ a left A-homomorphism, $K\colon M \to M/S$ the natural projection, then there exists $G\colon Az \to M$ such that $KG = F$, showing that ${}_AAz$ is projective, which yields $z = 0$, a contradiction. Thus $Z = 0$ and (4) implies (5) and (6) by [**8**]$_3$ (Theorem 4).

Assume (5). If $0 \neq b \in A$ such that $b^2 = 0$, the set of proper left subideals of Ab as a maximal member K by Zorn's Lemma, whence ${}_AAb/K$ is simple. If $g\colon Ab \to Ab/K$ is the natural projection, then there exists $c \in A$ such that $b + K = g(b) = bcb + K$. Now since A is left duo, $bc \in Ab$ implies $b \in K$, whence $Ab = K$, a contradiction. This proves that A is reduced. Then it may be proved that $Ad + l(d) = A$ for any $d \in A$ (because A is left duo) yielding A strongly regular and (5) implies (7).

Similarly, (6) implies (7) by [**4**] (Corollary 6). (7) implies (8) by Lemma 5(2).

Assume (8). Let M be a reduced maximal left ideal of A. If $0 \neq b \in A$ such that $b^2 = 0$, then $(Ab)^2 = 0$ implies $M \cap Ab = 0$, whence $A = M \oplus Ab$, contradicting $(Ab)^2 = 0$. This proves A reduced and (8) implies (1) by Lemma 5(1).

In view of Theorem 5.1, we may assert that quasi-injective modules need not be YJ-injective and the converse is not true either. Also, a quasi-injective YJ-injective module needs not be injective.

[**8**]$_6$ (Lemma 1) and the proof of Theorem 5.1 yield

PROPOSITION 6. *The following conditions are equivalent:*

(1) *A is left and right self-injective strongly regular with non-zero socle.*

(2) *A is a left duo ring containing a reduced injective maximal left ideal.*

(3) *A is a left duo ring containing an injective maximal left ideal M such that uM_A is YJ-injective for every $u \in M$.*

Proposition 6 and $[\mathbf{8}]_7$ (Lemma 7) motivate the next interesting remark.

Remark 5. If A is left self-injective containing a reduced maximal left ideal, then either A is regular with non-zero socle or strongly regular.

The proof of Theorem 5(1) (4) also yields.

Remark 6. A is left self-injective regular iff A is left self-injective such that the sum of any two injective left A-modules is YJ-injective.

Combining Lemmas 3 and 5 with Theorem 5.1, we get a few nice characteristic properties of commutative regular rings (cf. [**5**], p. 272).

Proposition 7. *If A is commutative, the following are then equivalent:*

(1) *A is regular.*
(2) *A is a YJ-injective ring whose principal ideals are flat.*
(3) *Every simple A-module is YJ-injective.*
(4) *Every maximal ideal of A is YJ-injective.*

The next «singular ideal intersection» property for rings whose simple right modules are either injective or projective is apparently new.

Remark 7. Consider the following statements: (a) Every simple right A-module is either injective or projective. (b) Every simple right A-module is either YJ-injective or projective. (c) $Y \cap Z = 0$. Then (a) implies (b) which, in turn, implies (c).

A is called *a left Kasch ring* iff every maximal left ideal of A is a left annihilator $[\mathbf{2}]_2$. Artinian Kasch rings are studied in [**6**].

Remark 8. The following conditions are equivalent for a ring A whose simple left modules are either injective or projective: (a) A is a left Kasch ring. (b) Every essential left ideal of A is a left annihilator (cfr. $[\mathbf{8}]_4$, Theorem 1).

Remark 9. (1) If A is a right Kasch ring, then $Z \subseteq J$. (2) If A is right YJ-injective such that every maximal left ideal is principal, then A is left Kasch.

The next result completes $[\mathbf{8}]_6$ (Theorem 11).

Theorem 8. *The following conditions are equivalent:*

(1) *A is semi-simple Artinian.*
(2) *A is a left Kasch ring whose simple right modules are YJ-injective.*

(3) *A is a left Kasch ring such that any minimal left ideal is YJ-injective.*

(4) *A is a left non-singular left YJ-injective left Σ-ring.*

(5) *A is a left non-singular left YJ-injective right Σ-ring.*

(6) *A is a left Σ-ring whose simple left modules are YJ-injective and flat.*

(7) *The right annihilator of any maximal left ideal of A is non-zero YJ-injective.*

(8) *A is a left YJ-injective left Kasch ring such that the sum of any two injective left A-modules is YJ-injective.*

Proof. Obviously, (1) implies (2) through (8).

Assume (2). Suppose there exists a non-zero ideal T such that $T^2 = 0$. If $0 \neq b \in T$, then $AbA + r(b) \neq A$. Let M be a maximal right ideal containing $AbA + r(b)$, $f: bA \to A/M$ the right A-homomorphism defined by $f(ba) = a + M$ for all $a \in A$. Then there exists $y \in A$ such that $1 + M = f(b) = yb + M$ which yields $1 \in M$, a contradiction. This proves A semi-prime and therefore (2) implies (1).

Assume (3). If $M = l(b)$ is a maximal left ideal of A, $b \in A$, then Ab is a minimal left ideal which is YJ-injective. Suppose that $(Ab)^2 = 0$. If i: $Ab \to Ab$ is the identity map, there exists $c \in A$ such that $b = i(b) = bcb$ which proves that Ab is generated by a non-zero idempotent, contradicting $(Ab)^2 = 0$. Therefore Ab is a direct summand of ${}_AA$, whence ${}_AA/M$ is projective, implying that ${}_AM$ is a direct summand of ${}_AA$. Thus (3) implies (1).

Either (4) or (5) implies (1) by $[2]_2$ (Corollary 5.13) and Remark 1(b).

Assume (6). Since every simple left A-module is YJ-injective and flat, then A is semi-prime such that every non-zero-divisor is invertible in A. Since A is left Goldie, then (6) implies (1).

Assume (7). Let M be a maximal left ideal, $0 \neq b \in r(M)$. There exists a positive integer n such that $b^n \neq 0$ and if $i: b^nA \to r(M)$ the inclusion map, then $b^n = i(b^n) = yb^n$ for some $y \in r(M)$. Now $M = l(y)$ and if $Ay \cap M \neq 0$, then Ay (being minimal) is contained in M which implies $b^n = yb^n = y^2b^n = 0$, a contradiction. Therefore $Ay \cap M = 0$, which proves that ${}_AM$ is a direct summand of ${}_AA$ and hence (7) implies (1).

Finally, Remark 1(a) and the proof of Theorem 1.5(4) show that (8) implies (1).

We conclude with a last remark.

Remark 10. (1) A is right hereditary iff every essential right ideal of A is either projective or a YJ-injective right annihilator. (2) A is simple

Artinian iff A is a simple right YJ-injective ring with a maximal right annihilator.

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Abstract

Generalizations of von Neumann regular and self-injective rings, quasi-Frobeniusean and Artinian rings are studied through HS-injectivity, Δ-rings, Σ-rings and Kasch rings. Conditions for classical quotient rings to be strongly regular and reduced Artinian are also given.

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