Riv. Mat. Univ. Parma (4) 11 (1985), 131-134

DĂNUŢ MARCU (*)

Note on the spanning trees of a connected graph (**)

Introduction

This paragraph is meant to present some definitions and results that are necessary to follow the further notes, our graph theoretic terminology being fairly standard [**1**], [**2**].

A *spanning tree* of a graph (an undirected graph) is a tree of the graph that contains all the vertices in the graph. It is well-known that a graph is connected if and only if contains a spanning tree [**1**].

A *cocycle* is a minimal set of edges in a graph the removal of which will increase the number of connected components in the remaining subgraph, whereas the removal of any its proper subset will not. It follows that in a connected graph the removal of a cocycle will separate the graph into two parts. This suggests an alternative way of defining a cocycle. Let the vertices in a connected component of a graph be divided into two subsets such that every two vertices in one subset are connected by a chain that contains only vertices in the subset. Then, the set of edges joining the vertices in the two subsets is a cocycle.

Since the removal of any edge e from a spanning tree T breaks the spanning tree up into two trees (that may consist of a single vertex), it follows that for every edge in a spanning tree there is an unique corresponding cocycle of the graph, called *fundamental cocycle* with respect to e and T. A graph is said to be obtained from G by *open-circuiting* and edge e, if it is obtained removing e from G. The open-circuiting of more than one edge is defined similarly (e.g., see [**3**]$_2$).

(*) Indirizzo: Faculty of Mathematics, University of Bucharest, Academiei 14, 70109 Bucharest, Romania.

(**) Ricevuto: 28-VI-1983.

A graph (possibly a multigraph) is said to be obtained from G by *short-circuiting* an edge e, if it is obtained by identifying, i.e., combining into one vertex, the two end-vertices of e and then removing e. The short-circuiting of more than one edge is defined similarly (e.g., see [**3**]$_2$).

If $T = \{e_1, e_2, \dots, e_m\}$ is a spanning tree of a connected graph G (according to [**1**], G contains $m+1$ vertices), we shall denote by $G(T, e_{i_1}, e_{i_2}, \dots, e_{i_k})$, $k = 2, 3, \dots, m$ the graph (possibly the multigraph) obtained from G by open-circuiting the edges $e_{i_1}, e_{i_2}, \dots, e_{i_{k-1}}$ and short-circuiting the edges $\{f_1, f_2, \dots, f_t\} = T - \{e_{i_1}, e_{i_2}, \dots, e_{i_k}\}$, $t = m - k$.

Remark. It is easy to check that the order, in which the above two graph's transformations are applied to obtain $G(T, e_{i_1}, e_{i_2}, \dots, e_{i_k})$, can be arbitrary.

The main results. Let $G = (V, E)$ be a connected graph (V is the set of vertices, and E the set of edges), T a spanning tree of G and $e \in T$. We shall denote by $C(e, T)$ the fundamental cocycle with respect to e and T.

Let $v \in V$ and $e \in E$, arbitrary chosen, such that e is incident with the vertex v. We denote by $C(v)$ the cocycle associated to the bipartition $(\{v\}, V - \{v\})$. Let $A(e)$ the set of spanning trees that contain the edge e and $\bar{A}(e)$ the set of spanning trees that do not contain e.

Lemma ([**3**]$_1$, Theorem 3).

$$\bar{A}(e) = \bigcup_{\substack{T \in A(e) \\ b \in C(e,T) \cap C(v) \\ b \neq e}} \{(T - \{e\}) \cup \{b\}\} .$$

Let now $T = \{e_1, e_2, \dots, e_m\}$ be a fixed spanning tree of a connected graph $G = (V, E)$, such that for every $k = 1, 2, \dots, m$ the subset $\{e_1, e_2, \dots, e_k\}$ forms a connected subgraph of G (obviously, the edges of a spanning tree can be always arranged in this way). Replacing the edges $e_{i_1}, e_{i_2}, \dots, e_{i_k}$ ($k \leqslant m$) of T with k distinct edges from $E - T$, we obtain a spanning subgraph S of G. If S is connected, then it is a spanning tree of G. So, let us denote by $A(T, e_{i_1}, e_{i_2}, \dots, e_{i_k})$ the set of the distinct spanning trees of G obtained by replacing, in all the possible ways, the edges $e_{i_1}, e_{i_2}, \dots, e_{i_k}$ of T with k distinct edges from $E - T$ (obviously, $A(T, e_{i_1}, e_{i_2}, \dots, e_{i_k})$ can be empty).

Theorem. *If* $1 \leqslant i_1 < i_2 < \dots < i_k \leqslant m$, *then*

$$A(T, e_{i_1}, e_{i_2}, \dots, e_{i_k}) = \bigcup_{\substack{T' \in A(T, e_{i_1}, e_{i_2}, \dots, e_{i_{k-1}}) \\ b \in C(e_{i_k}, T) \cap C(e_{i_k}, T') \\ b \neq e_{i_k}}} \{(T' - \{e_{i_k}\}) \cup \{b\}\} .$$

Proof. Let $\tilde{G} = G(T, e_{i_1}, e_{i_2}, \ldots, e_{i_k})$ and $T' \in A(T, e_{i_1}, e_{i_2}, \ldots, e_{i_{k-1}})$ arbitrary chosen. We denote by $\tilde{T}'$ what it remains from T' in $\tilde{G}$, i.e., $\tilde{T}' = T' - \{f_1, f_2, \ldots, f_t\}$. Obviously, since T' does not contain the edges $e_{i_1}, e_{i_2}, \ldots, e_{i_{k-1}}$, $\tilde{T}'$ is a spanning tree of $\tilde{G}$ (thus $\tilde{G}$ is connected), and $\tilde{T}' \in \tilde{A}(e_{i_k})$.

Because the set $\{e_1, e_2, \ldots, e_{i_k}\}$ forms a connected subgraph of T and $\{e_{i_1}, e_{i_2}, \ldots, e_{i_k}\} \subseteq \{e_1, e_2, \ldots, e_{i_k}\}$, it results that one of the two connected components of $T - \{e_{i_k}\}$ is either an isolated vertex (if e_{i_k} is incident with a pendant vertex of T) or a subgraph whose set of edges is contained in $T - \{e_{i_1}, e_{i_2}, \ldots, e_{i_k}\}$.

Let v be the vertex of this connected component of $T - \{e_{i_k}\}$ that is incident with e_{i_k} (this component is reduced to v in $\tilde{G}$). Obviously, the procedure for obtaining $\tilde{G}$ does not affect the fundamental cocycle $C(e_{i_k}, T)$ that coincides with $\tilde{C}(v)$ in $\tilde{G}$. On the other hand, the fundamental cocycles $\tilde{C}(e_{i_k}, \tilde{T}')$ and $C(e_{i_k}, T')$ differ one of another only by edges from the set $\{e_{i_1}, e_{i_2}, \ldots, e_{i_{k-1}}\}$, these edges being open-circuited. Thus, the following equality holds

$$\tilde{C}(v) \cap \tilde{C}(e_{i_k}, \tilde{T}') = C(e_{i_k}, T) \cap C(e_{i_k}, T') . \tag{1}$$

From (1) and Lemma (used for the connected graph $\tilde{G}$) it follows that

$$\bar{\tilde{A}}(e_{i_k}) = \bigcup_{\substack{\tilde{T}' \in \tilde{A}(e_{i_k}) \\ b \in C(e_{i_k}, T) \cap C(e_{i_k}, T') \\ b \neq e_{i_k}}} \{(\tilde{T}' - \{e_{i_k}\}) \cup \{b\}\} . \tag{2}$$

According to $[\mathbf{3}]_1$ (Theorem 4), $\tilde{T}'$ is uniquely obtained from T' by the construction of $\tilde{G}$. Since $\{f_1, f_2, \ldots, f_t\} \subseteq T'$ for each $T' \in A(T, e_{i_1}, e_{i_2}, \ldots, e_{i_{k-1}})$, then, by $[\mathbf{3}]_2$ (theorems 5.1 and 4.1), $T' := \tilde{T}' \cup \{f_1, f_2, \ldots, f_t\}$ is unique and belongs to $A(T, e_{i_1}, e_{i_2}, \ldots, e_{i_{k-1}})$ for every $\tilde{T}' \in \tilde{A}(e_{i_k})$.

Thus, the sets $\tilde{A}(e_{i_k})$ and $A(T, e_{i_1}, e_{i_2}, \ldots, e_{i_{k-1}})$ are in a bijective correspondence. Similarly, the sets $\bar{\tilde{A}}(e_{i_k})$ and $A(T, e_{i_1}, e_{i_2}, \ldots, e_{i_k})$. Hence, the theorem is proved according to (2).

Corollary. *If* $\{e_{i_1}, e_{i_2}, \ldots, e_{i_k}\}$ *and* $\{e_{j_1}, e_{j_2}, \ldots, e_{j_r}\}$ *are distinct subsets of* T, $k, r = 1, 2, \ldots, m$, *then* $A(T, e_{i_1}, e_{i_2}, \ldots, e_{i_k}) \cap A(T, e_{j_1}, e_{j_2}, \ldots, e_{j_r}) = \emptyset$.

Proof. It readily follows from theorem and the definition of a fundamental cocycle.

Remark. If T is a spanning tree of G an e and edge belonging to T, then a particular case of the above theorem is the following relation

$$A(T, e) = \bigcup_{\substack{b \in C(e, T) \\ b \neq e}} \{(T - \{e\}) \cup \{b\}\} .$$

Example. Let us consider the connected graph $G = (V, E)$ with $V = \{v_1, v_2, v_3, v_4\}$, $E = \{a = (v_1, v_2),\ b = (v_1, v_4),\ c = (v_1, v_3),\ d = (v_2, v_3),\ e = (v_2, v_4), f = (v_3, v_4)\}$ and $T = \{a, b, d\}$ a spanning tree of G for which $\{a\}$, $\{a, b\}$ and $\{a, b, d\}$ are connected subgraphs of G. We have

$$C(a, T) = \{a, c, e, f\}\ , \quad C(b, T) = \{b, e, f\}\ .$$

According to the above remark we obtain

$$A(T, a) = \{\{b, c, d\}, \{b, d, e\}, \{b, d, f\}\}\ .$$

On the other hand we have

$$C(b, \{b, c, d\}) = \{b, e, f\}\ , \quad C(b, \{b, d, f\}) = \{a, b, e\}\ , \quad C(b, \{b, d, f\}) = \{a, b, c\}\ ,$$

$$C(b, \{b, c, d\}) \cap C(b, T) = \{b, e, f\}\ , \quad C(b, \{b, d, e\}) \cap C(b, T) = \{b\}\ ,$$

$$C(b, \{b, d, f\}) \cap C(b, T) = \{b\}\ ,$$

and from the above theorem we obtain

$$A(T, a, b) = \{\{c, d, e\}, \{c, d, f\}\}.$$

Acknowledgement. I am grateful to the referee for his careful reading of the paper and for useful suggestions which led to better and more complete formulations of the stated material.

References

[1] C. BERGE, *Graphes et hypergraphes*, Dunod, Paris 1970.
[2] F. HARARY, *Graph theory*, Addison-Wesley, Reading Mass. 1969.
[3] D. MARCU: $[\cdot]_1$ *Note on the spanning trees of a connected digraph*, Review of Research Fac. Sci. Univ. Novi Sad **11** (1981), 297-303; $[\cdot]_2$ *On some vector spaces associated to a digraph*, Bull. Math. Soc. Sci Math. R. S. Roumanie (27) 4 (1983), 335-341.

Résumé

Une théorème, concernant les arbres partiels d'un graphe connexe, est donnée.

* * *