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Outlinks and inlinks in digraphs (**)

We study problems for digraphs analogous to graphs with constant link. The *outneighborhood* $N_0(v)$ of a point v of a digraph D is the set of all points of D adjacent from v. The *outlink of* v written $L_0(v)$ is the subdigraph of D induced by $N_0(v)$. If all the outlinks of the points of D are isomorphic to L_0, we say that *digraph* D *has outlink* L_0. Similarly we define the *in-neighborhood* $N^i(v)$, the *inlink of* v written $L^i(v)$ and the *inlink of a digraph* (if any).

In the case of graphs, it seems that Zykov [**6**] first asked which graphs L are *link graphs*, i.e., can be the (constant) link of some graph G. Brown and Connelly [**2**] showed that all paths, except P_3, and all cycles are link graphs. The case when L is a tree was studied by Blass, Harary and Miller [**1**] and by Hell [**5**]. Recently Hall [**3**] proved that there are exactly three graphs having the Petersen graph P as their link. For notation and terminology not given here, we refer to the book [**4**].

We now give some examples of finite and infinite digraphs with constant outlink or inlink. In particular we construct digraphs with nonisomorphic outlink and inlink. A digraph with both constant outlink and inlink has *constant links*.

The *dipath* $\boldsymbol{P}_n$ has n points. Thus in particular $\boldsymbol{P}_2$ is an oriented K_2. The infinite digraph D of Figure 1 has $\boldsymbol{P}_2$ as its outlink, but does not have a constant

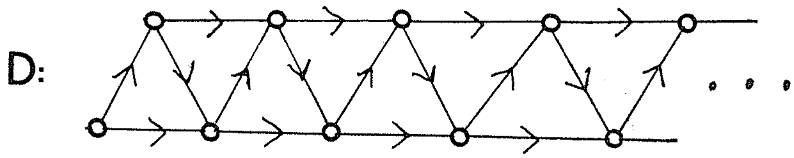

Figure 1

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inlink: the first point has no inlink as it is a transmitter, the second point has inlink K_1 and all other points have inlink $\boldsymbol{P}_2$. By directional duality, the converse D' has inlink $\boldsymbol{P}_2$ but no constant outlink.

On extending this D infinitely to the left as well, we get an infinite digraph with $\boldsymbol{P}_2$ as both its outlink and its inlink.

Question 1. Which digraphs are outlink digraphs of some finite digraph? of some infinite digraph? In particular which dicycles $\boldsymbol{C}_n$ are outlink digraphs? Which dipaths $\boldsymbol{P}_n$?

Whenever there is a digraph with outlink L_0, one can pose the extremal problem of constructing the smallest such digraphs. One digraph D_1 is *smaller* than another, D_2, if it has fewer points, $p_1 < p_2$. When $p_1 = p_2$, we regard D_1 as smaller than D_2 if it has fewer arcs, $q_1 < q_2$.

For example, the square of $\boldsymbol{C}_5$ is the smallest digraph having $\boldsymbol{P}_2$ as both its outlink and inlink.

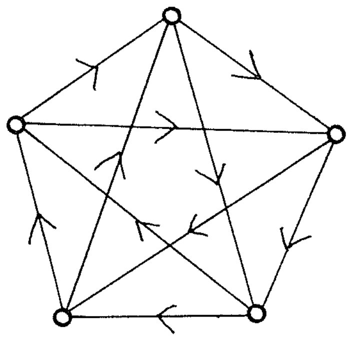

Figure 2

Obviously $\boldsymbol{C}_7^3$ has the transitive triple T_3 as both its outlink and inlink. Similarly every transitive tournament (linear order) T_n is both an outlink and an inlink.

Similar questions suggest themselves concerning outstars and instars.

An outstar An instar

Figure 3

Question 2. Which ordered pairs of digraphs (E_1, E_2) can occur as the outlink E_1 and the inlink E_2 of some digraph D?

We now display some examples of digraphs with different outlink and inlink. Begin with the usual covering of the plane by equilateral triangles.

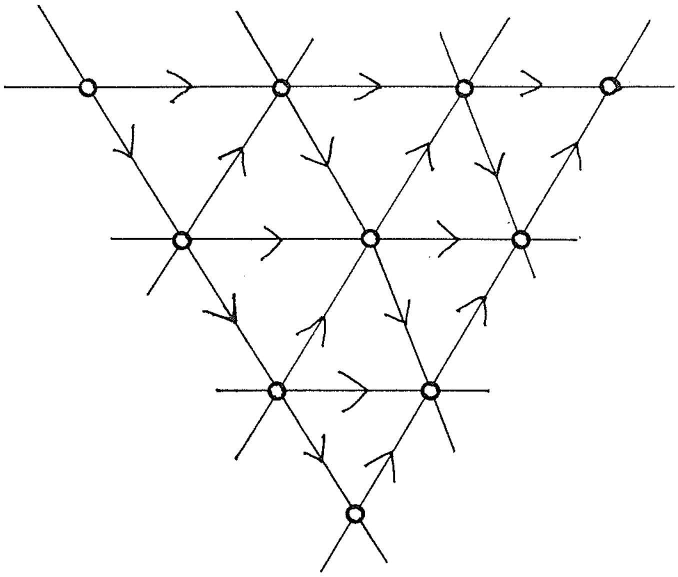

Figure 4

Then orient every edge from left to right. The result is an infinite digraph with outlink and inlink which are converse semipaths. A finite digraph is obtained from this by « rolling it up » to form a torus. The smallest such digraph so constructed has 9 points.

The second example of a digraph with different outlink and inlink is interesting as these do not constitute a converse pair. The digraph is shown in Figure 5 with its outlink and inlink.

The phenomenon observed in this example is general, namely, in every finite digraph D with constant links, the number $q(L_0)$ of arcs in the outlink of D equals the number $q(L^i)$ in its inlink. As noted by J. Hall, both numbers equal the ratio of the number of transitive triples in D to $p(D)$, the number of points.

If D is permitted to be infinite, then it may have constant outlink and constant inlink with different numbers of arcs, which confirms a conjecture of J. Hall. For example, let the points of D be the infinite sequences $v = (v_0, v_1, v_2, \ldots)$ all of whose entries are 0 or 1, and let there be arcs from any such v to the three points $(0, v_0, v_1, \ldots)$, $(1, v_0, v_1, \ldots)$, and $(1 - v_0, v_1, v_2, \ldots)$.

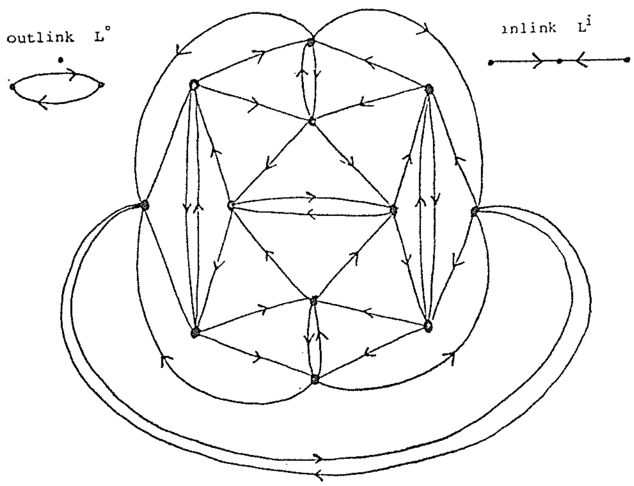

Figure 5

Then the outlink of D is a symmetric pair plus an isolated point while the inlink is an arc. In this digraph, the numbers of arcs in the links are 2 and 1.

The smallest oriented graph with different outlink and inlink has eight points and is unique. Using the convenient notation of Roberto Frucht, this digraph can be written as **8** (1, 2, 5); a fragment is shown in Figure 6. The outlink and inlink are the same as for the 9-point oriented graph obtained from Figure 4 by putting it on a torus.

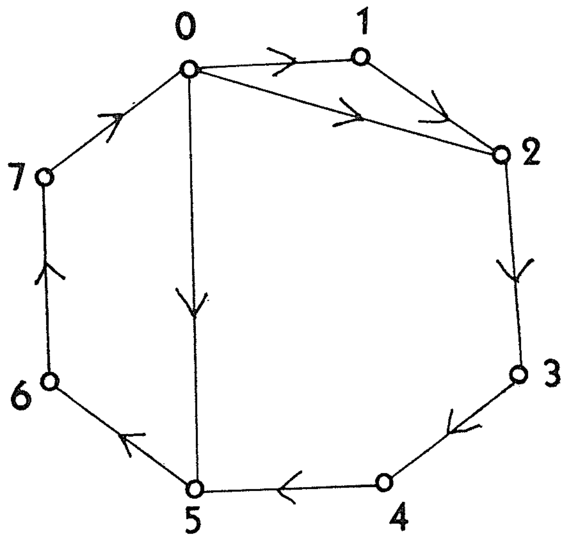

Figure 6

There is also a unique smallest digraph with different outlink and inlink. It has just six points and is a mixed graph with three undirected lines (standing for symmetric pairs) and twelve arcs as shown in Figure 7, with its outlink and inlink.

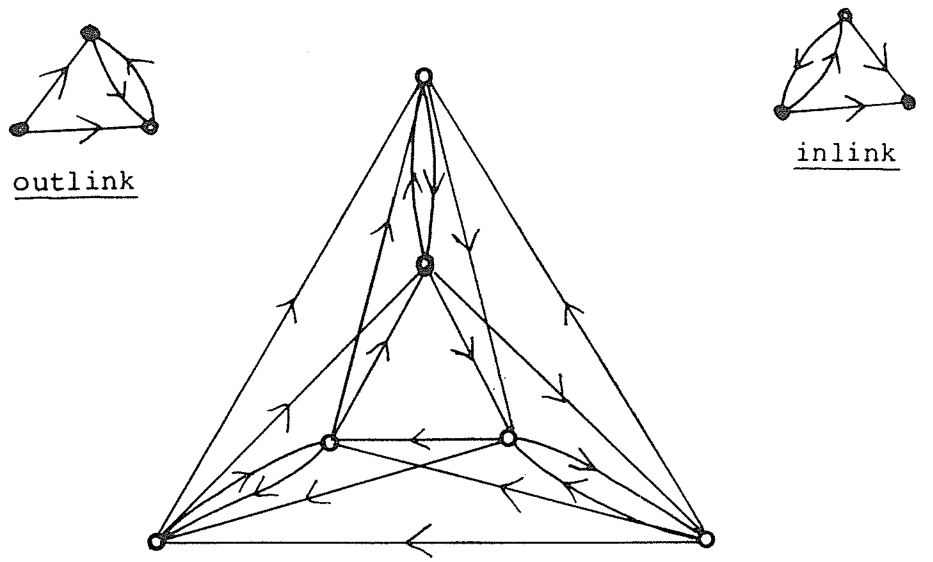

Figure 7

Of course there is a digraph with constant outlink that does not have constant inlink. The smallest such digraph is very small, as is the smallest oriented graph (Figure 8).

Figure 8

There is a digraph D with both constant outlink and constant inlink whose underlying graph does not have constant link. An example is shown in Figure 9, but we do not know if it is as small as possible.

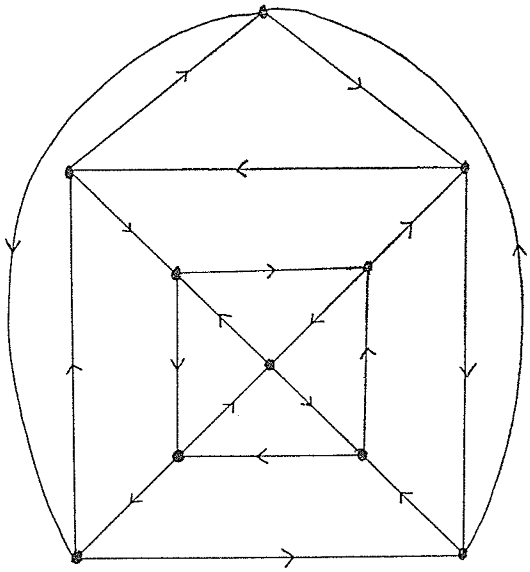

Figure 9

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Abstract

The link of a point of an undirected graph is the subgraph induced by its neighborhood. In a graph with constant link, all the links are isomorphic. Then a link graph is the constant link of some graph. The corresponding concepts are now introduced for digraphs in which each point has an outneighborhood and an in-neighborhood, namely: the outlink and inlink of a point, a digraph with constant outlink or inlink, and an outlink (inlink) digraph. Examples are given of both finite and infinite digraphs with constant outlink or inlink. In particular we construct the smallest digraph with different constant outlink and inlink, and a digraph with both constant links whose underlying graph does not have constant link. Several related unsolved problems are presented.

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