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Transmission matrix for thermal oscillations trough walls, spinor calculus and ill-posed problems (**)

1 - Introduction

We take here a wall of finite unknown thickness b and constant unknown density ϱ, whose boundaries are kept to prescribed sinusoidal temperatures $t_1 e^{i\omega\tau} \cdot t_2 e^{i\omega\tau}$ (t_1, t_2 complex numbers): here and below τ will denote time and t will denote temperature. We also suppose that the corresponding heat fluxes $\Phi_1 e^{i\omega\tau}$, $\Phi_2 e^{i\omega\tau}$ are given (Φ_1, Φ_2 complex numbers).

By the knowledge of the heat fluxes and of boundary temperatures, we propose to determine thickness, thermal conductivity or density of the wall, or, anyway, a couple of quantities arbitrarely choosen between thickness, thermal conductivity, and ϱc, the product of density and specific heat of the wall.

The problem we are given clearly belongs to the class of ill-posed problems in partial derivative equations, so it will expected that the solutions (if any) will exhibite a sort of instability, depending on the precision of the measurement of t_1, t_2, Φ_1, Φ_2. Similar problems have been largely studied elsewere (see e.g. [1], [2], [3]).

We treat here the particularly simple case of a thermal oscillation of a given frequency $\omega/2\pi$, wich propagates uniformely in a wall, and with a given change in phase. This problem is studied at a some extent in [4]. In this

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case, it is very easy to relate the values t_1, Φ_1 of the temperature and the flux at the initial abscissa $x = 0$ of the wall, to the values t_2, Φ_2 at the final abscissa $x = b$. The connection is given by the transmission matrix (see **2** below) which is a 2×2 complex unimodular (but not unitary) matrix. This matrix so allow us to determine t_2, Φ_2 because to the fact that its elements are known complex numbers. The inverse problem is then to determine the elements of the transmission matrix by supposing that the quantities t_1, Φ_1, t_2, Φ_2 are fixed a priori. This problem is solved in **2**, while **3** is devoted to the analysis of the compatibility of the solutions. As the transmission matrix assumes so great relevance for our problem, it is natural to study its more important features. We adopt here a point of view not usual in this context, namely a group representation analysis. Then, in **4** we explicitely point out that the transmission matrix, which is an element of C_2, the group of 2×2 unimodular matrices, corresponds to a particular Lorentz transformation. So we can use the result of spinor calculus in writing down its properties. In **5** we briefly discuss the illness of the problem.

2 - The elements of the transmission matrix

We propose ourselves to determine the diffusivity a_t and the thickness b of a wall, by only measuring temperatures and heat fluxes. As usual we suppose that one surface of the wall be coincident, in a xyz system, with the yz plane, and that the interior part of the wall is made of an homogeneous medium of diffusivity $a_t = k/\varrho c$, k (constant) being the thermal conductivity, ϱ (constant) the density, and c (constant) the specific heat of the said medium. Let we place the other surface at the plane $x = b$. Let we denote the temperature inside the wall by t, and let τ be the time.

From the symmetry of the problem it is clear that the temperature t is independent of y and z, so the diffusion equation reads, t being a x-C^2, τ-C^1 function of x and τ

$$t_{xx} = \frac{1}{a_t} t_\tau .$$

The boundary conditions to be fulfilled are

$$t(0, \tau) = t_1 e^{i\omega\tau}, \qquad t(b, \tau) = t_2 e^{i\omega\tau},$$

where t_1, t_2 are complex numbers. This means that we limit ourselves to the case of a thermal oscillation of a given frequency $\omega/2\pi$, and that we fix the relative phases of the ingoing and of the outgoing temperatures.

The solution we look for has the form

$$t(x, \tau) = \Theta(x)\, e^{i\omega\tau},$$

then the equation to be satisfied by $\Theta(x)$ is $\Theta'' - (i\omega/a_t)\Theta = 0$. We have

$$\Theta(x) = \frac{t_2 \operatorname{Sh} x\sqrt{i\omega/a_t} + t_1 \operatorname{Sh}(b-x)\sqrt{i\omega/a_t}}{\operatorname{Sh} b\sqrt{i\omega/a_t}} \qquad 0 \leqslant x \leqslant b,$$

which satisfies the conditions $\Theta(0) = t_1$, $\Theta(b) = t_2$. Then $t(x, \tau) = \Theta(x)\, e^{i\omega\tau}$ is the solution which fulfills the requested boundary conditions.

For the corresponding thermal flux we get

$$\varphi(x, \tau) = -k\,\frac{\partial t(x, \tau)}{\partial x} = \Phi(x)\exp(i\omega\tau),$$

with
$$\Phi(x) = -k\sqrt{i\omega/a_t}\,\frac{t_2 \operatorname{Ch} x\sqrt{i\omega/a_t} - t_1 \operatorname{Ch}(b-x)\sqrt{i\omega/a_t}}{\operatorname{Sh} b\sqrt{i\omega/a_t}},$$

from which we obtain

$$\Phi(0) = \Phi_1 = -k\sqrt{i\omega/a_t}\,\frac{t_2 - t_1 \operatorname{Ch} b\sqrt{i\omega/a_t}}{\operatorname{Sh} b\sqrt{i\omega/a_t}},$$

$$\Phi(b) = \Phi_2 = -k\sqrt{i\omega/a_t}\,\frac{t_2 \operatorname{Ch} b\sqrt{i\omega/a_t} - t_1}{\operatorname{Sh} b\sqrt{i\omega/a_t}}.$$

The link between t_1, Φ_1 and t_2, Φ_2 is given by

$$(1)\quad \begin{pmatrix} t_2 \\ \Phi_2 \end{pmatrix} = T\begin{pmatrix} t_1 \\ \Phi_1 \end{pmatrix} = \begin{pmatrix} \operatorname{Ch} b\sqrt{i\omega/a_t} & -(1/k)\sqrt{a_t/i\omega}\operatorname{Sh} b\sqrt{i\omega/a_t} \\ -k\sqrt{i\omega/a_t}\operatorname{Sh} b\sqrt{i\omega/a_t} & \operatorname{Ch} b\sqrt{i\omega/a_t} \end{pmatrix}\begin{pmatrix} t_1 \\ \Phi_1 \end{pmatrix}.$$

The 2×2 unimodular complex matrix T is the transmission matrix, which, by posing, as usual, $y = b\sqrt{\omega/2a_t}$, $w = k\sqrt{\omega/2a_t}$, reads (y positive real pure number, w positive real number having the physical dimension of a flux/degree)

$$T = \begin{pmatrix} \operatorname{Ch}(1+i)y & -(1-i)/2w \operatorname{Sh}(1+i)y \\ -(1+i)w\operatorname{Sh}(1+i)y & \operatorname{Ch}(1+i)y \end{pmatrix}.$$

The eigenvalues of T are $\lambda_1, \lambda_2 = \operatorname{Ch}(1+i)y \pm \operatorname{Sh}(1+i)y$, so the diagonal form of T is given by

$$T_d = \begin{pmatrix} \exp(1+i)y & 0 \\ 0 & \exp(-(1+i)y) \end{pmatrix}, \qquad \text{and the eigenspaces are}$$

$$H_{\lambda_1} = \begin{pmatrix} \alpha \\ -(1+i)w\alpha \end{pmatrix}, \qquad H_{\lambda_2} = \begin{pmatrix} \beta \\ (1+i)w\beta \end{pmatrix} \quad (\alpha, \beta \in \mathscr{C}).$$

The diagonalizing matrix is then

$$P = \begin{pmatrix} 1 & 1 \\ -(1+i)w & (1+i)w \end{pmatrix} \quad \text{whose inverse is } P^{-1} = \tfrac{1}{2}\begin{pmatrix} 1 & -(1-i)/2w \\ 1 & (1-i)/2w \end{pmatrix}.$$

So we have $\begin{pmatrix} t_2 \\ \Phi_2 \end{pmatrix} = T\begin{pmatrix} t_1 \\ \Phi_1 \end{pmatrix} = PP^{-1}TPP^{-1}\begin{pmatrix} t_1 \\ \Phi_1 \end{pmatrix} = PT_dP^{-1}\begin{pmatrix} t_1 \\ \Phi_1 \end{pmatrix}$,

or in other words $2wt_2 \mp (1-i)\Phi_2 = e^{\pm(1+i)y}\left(2wt_1 \pm (1-i)\Phi_1\right)$.

By some algebraic manipulations, we deduce from these equations, under the obvious assumption that $t_1\Phi_1 + t_2\Phi_2 \neq 0$,

$$\text{(2)} \quad \begin{gathered} \mathrm{Ch}\,(1+i)y = \frac{t_1\Phi_1 + t_2\Phi_2}{t_1\Phi_2 + t_2\Phi_1} = z_1\,, \qquad \frac{1-i}{2w}\,\mathrm{Sh}\,(1+i)y = \frac{t_1^2 - t_2^2}{t_1\Phi_2 + t_2\Phi_1} = z_2\,, \\ (1+i)w\,\mathrm{Sh}\,(1+i)y = \frac{\Phi_1^2 - \Phi_2^2}{t_1\Phi_2 + t_2\Phi_1} = z_3 \qquad \text{with } z_1^2 - z_2 z_3 = 1\,. \end{gathered}$$

We take all the numbers z_1, z_2, z_3 different from zero, for physical reasons (y and w are both $\neq 0$).

The system (2) can be also deduced by solving system (1) by respect to the unknowns y, w, instead of determining t_2, Φ_2 from t_1, Φ_1 given; in this way, we consider t_1, Φ_1, t_2, Φ_2 to be the known quantities.

Solutions of (2) are

$$\text{(3)} \quad \begin{gathered} y = (1/2)\cos^{-1}\left(|z_1|^2 - |z_1^2 - 1|\right) = (1/2)\,\mathrm{Ch}^{-1}\left(|z_1|^2 + |z_1^2 - 1|\right), \\ w^2 = -(i/2)\,z_3/z_2\,, \end{gathered}$$

which can be easily obtained from elementary equalities holding for hyperbolic functions (e.g. $|z_1|^2 = |\mathrm{Ch}\,(1+i)y|^2 = (1/2)(\mathrm{Ch}\,2y + \cos 2y)$, $|z_1^2 - 1| = |\mathrm{Sh}\,(1+i)y|^2 = (1/2)(\mathrm{Ch}\,2y - \cos 2y)$).

It is quite obvious that the last formulae define operators with domain and range different from $\mathscr{C}$. The next paragraph is then devoted to discuss domain and range of these operators.

3 - Analysis of solutions

We begin to observe that for any complex number z_1, we always have $|z_1|^2 - |z_1^2 - 1| \leqslant 1$. Moreover in our case $|z_1|^2 > 1$, as $\cos^2 y + \mathrm{Sh}^2 y = |z_1|^2$. We must now satisfy the condition

$$\cos^{-1}\left(|z_1|^2 - |z_1^2 - 1|\right) = \mathrm{Ch}^{-1}\left(|z_1|^2 + |z_1^2 - 1|\right).$$

This is achieved by noting that from $\cos\theta = |z_1|^2 - |z_1^2 - 1|$ we have $y = \theta/2 + k\pi = (1/2)\,\mathrm{Ch}^{-1}\left(|z_1|^2 + |z_1^2 - 1|\right)$, $(k \in N)$, if and only if

$$z_1 = \mathrm{Ch}\,(1 + i)(\theta/2 - k\pi).$$

For example, let $|z_1|^2 - |z_1^2 - 1|$ be equal to 1. Then $\theta = 2k\pi$, $y = k\pi$, and the admissible z_1 are of the form $z_1 = (-1)^k\,\mathrm{Ch}\,k\pi$ $(k \neq 0,\ k \in N)$. If $|z_1|^2 - |z_1^2 - 1| = -1$, $y = \pi/2 + k\pi$, and the admissible z_1 are of the form

$$z_1 = (-1)^k i\,\mathrm{Sh}\,(\pi/2 + k\pi) \qquad k \in N.$$

Let us now discuss $w^2 = -iz_3/(2z_2)$. In this case, we are compelled to suppose that $\mathrm{Re}\,z_3/z_2 = 0$, $\mathrm{Im}\,z_3/z_2 > 0$, otherwise w would not be a positive real number.

Summing up, the quantities t_1, Φ_1, t_2, Φ_2 cannot be assigned in a arbitrary way. They must fulfill the following bounds and conditions

$$t_1\Phi_2 + t_2\Phi_1 \neq 0, \qquad \left(\left|\frac{t_1\Phi_1 + t_2\Phi_2}{t_1\Phi_2 + t_2\Phi_1}\right|^2 - \left|\left(\frac{t_1\Phi_1 + t_2\Phi_2}{t_1\Phi_2 + t_2\Phi_1}\right)^2 - 1\right|\right) < 1,$$

$$\left|\frac{t_1\Phi_1 + t_2\Phi_2}{t_1\Phi_2 + t_2\Phi_1}\right|^2 > 1, \qquad \frac{t_1\Phi_1 + t_2\Phi_2}{t_1\Phi_2 + t_2\Phi_1} = \mathrm{Ch}\,(1 + i)(\theta/2 + k\pi) \qquad 0 \leqslant \theta \leqslant 2\pi,$$

$$\mathrm{Re}\,(\Phi_1^2 - \Phi_2^2)\,\mathrm{Re}\,(t_1^2 - t_2^2) + \mathrm{Im}\,(\Phi_1^2 - \Phi_2^2)\,\mathrm{Im}\,(t_1^2 - t_2^2) = 0,$$

$$\frac{\mathrm{Im}\,(\Phi_2^2 - \Phi_1^2)\,\mathrm{Re}\,(t_2^2 - t_1^2) - \mathrm{Re}\,(\Phi_2^2 - \Phi_1^2)\,\mathrm{Im}\,(t_2^2 - t_1^2)}{[\mathrm{Re}\,(t_2^2 - t_1^2)]^2 + [\mathrm{Im}\,(t_2^2 - t_1^2)]^2} > 0.$$

In this case, and only in this case, solutions of the problem are

$$b\sqrt{\frac{\omega}{2a_t}} = \frac{\theta}{2} + k\pi = \frac{1}{2}\cos^{-1}\left(\left|\frac{t_1\Phi_1 + t_2\Phi_2}{t_1\Phi_2 + t_2\Phi_1}\right|^2 - \left|\left(\frac{t_1\Phi_1 + t_2\Phi_2}{t_2\Phi_1 + t_1\Phi_2}\right)^2 - 1\right|\right) + k\pi,$$

$$k\sqrt{\frac{\omega}{2a_t}} = \frac{\sqrt{2}}{2}\sqrt{\mathrm{Im}\,\frac{z_3}{z_2}} = \frac{\sqrt{2}}{2}\sqrt{\frac{\mathrm{Im}\,(\Phi_2^2 - \Phi_1^2)\,\mathrm{Re}\,(t_2^2 - t_1^2) - \mathrm{Re}\,(\Phi_2^2 - \Phi_1^2)\,\mathrm{Im}\,(t_2^2 - t_1^2)}{[\mathrm{Re}\,(t_2^2 - t_1^2)]^2 + [\mathrm{Im}\,(t_2^2 - t_1^2)]^2}}.$$

4 - Spinors

Let us now observe that system (1) expresses the fact that the boundary values for t and Φ transforms exactly as spinors belonging to the $D_{1/2,0}$ representation of the Lorentz proper group L_p. In effect, the transmission matrix is an element of C_2, the group of the 2×2 complex unimodular matrices, and it is well know that L_p is homomorphic to C_2 (see e.g. [**6**]). The elements of the group C_2 are clearly characterised by six real parameters (one relation, the determinant equal to 1, holds between four complex elements). This is a remark which will be useful for future developements; in particular if a multi-layer wall is considered we already know that its transmission matrix will at most depend on six real parameters, no matter the number of the layers be larger than three.

We consider now, for dimensional homogeneity reasons, the column vector

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} t_1 - \frac{1}{2}\frac{1-i}{w}\Phi_1 \\ t_1 + \frac{1}{2}\frac{1-i}{w}\Phi_1 \end{pmatrix}$$

and we call it *thermal spinor* for the surface 1 of the given wall. The corresponding thermal spinor for the surface 2 will be denoted by u', with expression

$$u' = \begin{pmatrix} u'_1 \\ u'_2 \end{pmatrix} = \begin{pmatrix} t_2 - \frac{1}{2}\frac{1-i}{w}\Phi_2 \\ t_2 + \frac{1}{2}\frac{1-i}{w}\Phi_2 \end{pmatrix}.$$

In this manner the transmission matrix becomes diagonal (we called it T_d in **2**) and its elements are adimensional.

Let now we consider two different thermal spinors belonging to the same surface 1. We denote them with expressions

$$u = \begin{pmatrix} t_1^{(1)} - \frac{1}{2}\frac{1-i}{w}\Phi_1^{(1)} \\ t_1^{(1)} + \frac{1}{2}\frac{1-i}{w}\Phi_1^{(1)} \end{pmatrix}, \qquad v = \begin{pmatrix} t_1^{(2)} - \frac{1}{2}\frac{1-i}{w}\Phi_1^{(2)} \\ t_1^{(2)} + \frac{1}{2}\frac{1-i}{w}\Phi_1^{(2)} \end{pmatrix}.$$

The corresponding thermal spinors on surface 2 of the wall will be denoted with

$$u' = \begin{pmatrix} t_2^{(1)} - \frac{1}{2}\frac{1-i}{w}\Phi_2^{(1)} \\ t_2^{(1)} + \frac{1}{2}\frac{1-i}{w}\Phi_2^{(1)} \end{pmatrix}, \qquad v' = \begin{pmatrix} t_2^{(2)} - \frac{1}{2}\frac{1-i}{w}\Phi_2^{(2)} \\ t_2^{(2)} + \frac{1}{2}\frac{1-i}{w}\Phi_2^{(2)} \end{pmatrix}.$$

Due to the unimodularity of the transmission matrix T_d, one finds by direct calculation that

$$(t_2^{(1)} - \frac{1}{2}\frac{1-i}{w}\Phi_2^{(1)})(t_2^{(2)} + \frac{1}{2}\frac{1-i}{w}\Phi_2^{(2)}) - (t_2^{(2)} - \frac{1}{2}\frac{1-i}{w}\Phi_2^{(2)})(t_2^{(1)} + \frac{1}{2}\frac{1-i}{w}\Phi_2^{(1)})$$

$$= (t_1^{(1)} - \frac{1}{2}\frac{1-i}{w}\Phi_1^{(1)})(t_1^{(2)} + \frac{1}{2}\frac{1-i}{w}\Phi_1^{(2)}) - (t_1^{(2)} - \frac{1}{2}\frac{1-i}{w}\Phi_1^{(2)})(t_1^{(1)} + \frac{1}{2}\frac{1-i}{w}\Phi_1^{(1)})$$

= invariant, or in other words

$$\Phi_2^{(1)} t_2^{(2)} - \Phi_2^{(2)} t_2^{(1)} = \Phi_1^{(1)} t_1^{(2)} - \Phi_1^{(2)} t_1^{(1)} = \text{invariant.}$$

This basic invariant is nothing else that the scalar product of uv, which, with the notation $u^a = \varepsilon^{ab} u_b$ ($\varepsilon^{12} = -\varepsilon^{21} = 1$, $\varepsilon^{11} = \varepsilon^{22} = 0$ summation understood on the repeated index) can be written $u^a v_a$. As the complex conjugate of a spinor gives a «*dotted*» *spinor*, even the quantity $u^{\dot a} v_{\dot a}$ is an invariant ($u^{\dot a} = \varepsilon^{\dot a \dot b} u_{\dot b}$, $\varepsilon^{\dot 1 \dot 2} = -\varepsilon^{\dot 2 \dot 1} = 1$, $\varepsilon^{\dot 1 \dot 1} = \varepsilon^{\dot 2 \dot 2} = 0$).

As to every four vector $x^\mu \equiv (x^1, x^2, x^3, x^0)$ corresponds a spinor $u_{\lambda\dot\delta}$, and conversely, it is a standard procedure to write down the Lorentz transformation on x^μ which maps in the matrix T_d. In this manner, we see that the correspondance between a rank-two spinor of the $D_{1/2,1/2}$ representation of the proper Lorentz group L_p can be given in the following way

$$u_{1\dot 1} = x^3 + x^0, \quad u_{1\dot 2} = x^1 - ix^2, \quad u_{2\dot 1} = x^1 + ix^2, \quad u_{2\dot 2} = -x^3 + x^0.$$

Then the element of L_p which corresponds to T_d is the following

$$L_t = \begin{pmatrix} \cos 2y & \sin 2y & 0 & 0 \\ -\sin 2y & \cos 2y & 0 & 0 \\ 0 & 0 & \mathrm{Ch}\, 2y & \mathrm{Sh}\, 2y \\ 0 & 0 & \mathrm{Sh}\, 2y & \mathrm{Ch}\, 2y \end{pmatrix},$$

namely a proper rotation of amount $2y$ around the x^3-axis, together with a special Lorentz transformation along the x^3-axis, with velocity $\beta = \mathrm{Th}\, 2y$.

At this point, and only from a purely theoretical point of view, it would be tempting to write down a propagation equation for the thermal spinor field, which, in this context, could be called *thermon field*. The most simple covariant thermon equation we can guess is the following

$$\partial_{a\dot b} u^a = \frac{1}{\lambda} u_{\dot b},$$

with $\partial_{1\dot 1} = \partial^3 + \partial^0$, $\partial_{1\dot 2} = \partial^1 - i\partial^2$, $\partial_{2\dot 1} = \partial^1 + i\partial^2$, $\partial_{2\dot 2} = -\partial^3 + \partial^0$, $1/\lambda = \sqrt{\omega/2a_t}$.

As a first consequence we see that every component of u satisfies the Klein-Gordon equation $(\Box - 1/\lambda^2)\Phi = 0$ and so we would describe neutral particles (*thermons*) of rest-mass $m_0 = (\hbar/c)(\omega/2a_t)$ and spin 1/2.

With such a choice, we point out that the spinor equation would be not linear, as the operator $\partial_{a\bar{b}}$ on u acts as a multiple of the complex conjugation.

This point of view would be on the same line of the description of achoustical properties of materials in terms of phonons; however, we do not develope further the treatment, as, at the present state of the facts, no evidence for such thermons is known or needed.

5 - The problem as an ill-posed problem

The problem we have solved in **2** is clearly an inverse problem: we contend to determine y and w (which clearly contain the phisical quantities of interest a_t, b, K), from the knowledge of t_1, t_2, Φ_1, Φ_2. At this purpose, we know that the most important requisite for the solution of an inverse problem is its continuous dependance on the data we are given. This continuous dependance is often taken as a synonimous of stability [**5**].

Anyway, a qualitative criterium for a problem to be ill-posed, is the fact that a little error on data produces a great error on the solutions, as some classical examples show.

A more quantitative criterium is to assume that the solutions of the problem belong to a specific function space X (e.g. a scalar product space), and that the data belong to another function space Y (e.g. a scalar product space). In this way, if $x \in X$, $y \in Y$, there must exist an operator A which determines the solution of the direct problem through the relation $Ax = y$. The inverse problem is the to determine x such that $Ax = y$ whenever $y \in Y$ is given, or, in other words, to invert the operator A. The most common situation is that R_A, the range of A, does not coincide with Y, the data space. It is very evident that the solution does not exist if the datum y has a nonzero component which is orthogonal to R_A. Moreover, the solution is not uniquely determined if A is not injective.

Let us now proceed to discuss our specific case (first solution), for which X, Y are both $\mathcal{C}$, the complex field (equipped with the scalar product $(z_1, z_2) = z_1^* z_2$), A is given by the relation $\mathrm{Ch}\,(1+i)y = z_1$, and A^{-1} is given by the relation: $y = \frac{1}{2}\cos^{-1}(|z_1|^2 - |z_1^2 - 1|)$.

We have (see **3** above)

$$D_A = \{y \in \mathcal{C} \,|\, y \in R_+\},$$
$$R_A = \{z_1 \in \mathcal{C} \,|\, |z_1| > 1 \cap |z_1|^2 = \tfrac{1}{2}(\mathrm{Ch}\,2y + \cos 2y),\ y \in R_+\},$$
$$D_{A^{-1}} = \{z_1 \,|\, |z_1| > 1 \cap z_1 = \mathrm{Ch}\,(1+i)(\theta/2 + k\pi),\ 0 \leqslant \theta \leqslant \pi\},$$
$$R_{A^{-1}} = \{y \in \mathcal{C} \,|\, Y \in R_+\}.$$

We observe that A is not injective. Moreover we have that R_A does not coincide with the data space Y, which is $\mathcal{C}$. These are precisely the conditions under which we can say that the problem is an ill-posed problem.

The same argument holds for the second solution $2iw^2 = z_3/z_2$, for which, if we call B the mapping $B: w \to (z_2, z_3)$, we have

$$D_B = \{w \in C \mid w \in R_+\}\,, \qquad R_B = \{(z_2, z_3) \in \mathcal{C}^2 \mid \operatorname{Re} z_2/z_3 = 0,\ \operatorname{Im} z_2/z_3 > 0\}\,.$$

As B^{-1} is given by $w = \sqrt{2}/2\sqrt{\operatorname{Im} z_3/z_2}$ we have

$$R_{B^{-1}} = \{w \in \mathcal{C} \mid w \in R_+\}\,, \qquad D_{B^{-1}} = \{(z_2, z_3) \in \mathcal{C}^2 \mid \operatorname{Im} z_3/z_2 > 0\}\,.$$

Exactly as above, B is not injective and its range does not coincide with the data space, which in this case is $\mathcal{C} \times \mathcal{C}$.

We remark explicitely that the mappings A and B above are not linear ones, so many of the known results can not be applied here.

We can now proceed to study the rate of change of solutions depending on the variation of data. The more simple way to do this job is to examine the expressions for the absolute value of derivatives of y and w

$$|\Delta y| = \frac{|\Delta z_1|}{\sqrt{|z_1^2 - 1|}}\,, \quad |\Delta w| = \frac{1}{2}\,\frac{1}{|z_2|}\,\frac{1}{\sqrt{|z_2 z_3|}}\,|z_3 \Delta z_2 - z_2 \Delta z_3|\,.$$

From these formula we deduce that, at least when we move in a definite branch of $\ln(z_1^2 + \sqrt{z_1^2 - 1})$, no problem on continuity of solutions arises, except for neighborhoods of the point $z_1 = 1$. In fact, near this point we have that even small errors on z_1 imply very large errors on y. Analogous considerations hold for errors on w.

We observe now that, as the possible values for the quantity z are given by $z_{1,k} = \operatorname{Ch}(1+i)(\theta/2 + k\pi)$ $k \in N$, the absolue value of the difference between two contiguous values $z_{1,k}$, $z_{1,k+1}$ is given by

$$|z_{1,k} - z_{1,k+1}| = |(-1)^k\, 2 \operatorname{Ch} \pi/2 \operatorname{Ch}[(1+i)\,\theta/2 + (k + \tfrac{1}{2})\pi]$$

$$= 2 \operatorname{Ch} \pi/2 \sqrt{\cos^2 \theta/2 + \operatorname{Sh}^2(\theta/2 + (k + \tfrac{1}{2})\pi)}\,.$$

This absolute value goes to infinity of the order of k whenever k goes to infinity; nowithstanding that, the absolute value of the difference of the corresponding solutions y_k, y_{k+1} is bounded and has the constant value π. We could in this manner give another characterization of our ill-posed problem.

At least, we briefly discuss the behaviour of our solution $\theta(x)$, considered as a map $\theta: z_1 \to z$, where

$$z = \theta(z_1) = t_2 \frac{(z_1 + \sqrt{z_1^2 - 1})^{x/b} - (z_1 - \sqrt{z_1^2 - 1})^{x/b}}{2\sqrt{z_1^2 - 1}} + t_1 \frac{(z_1 + \sqrt{z_1^2 - 1})^{(b-x)/b} - (z_1 - \sqrt{z_1^2 - 1})^{(b-x)/b}}{2\sqrt{z_1^2 - 1}} .$$

Let we exhamine the variation of θ whenever z_1 varies of a little quantity near a value z_1^0.

Writing $|\theta(z_1) - \theta(z_1^0)| \sim |\mathrm{d}f/\mathrm{d}z_1|_{z_1 = z_1^0} |\Delta z_1|$, we have, moving ourselves in a definite branch, and after some tedious calculations

$$\frac{\mathrm{d}\theta}{\mathrm{d}z_1} = \frac{1}{2(z^2 - 1)} \{ t_2 [\frac{x}{b} [(z_1 + \sqrt{z_1^2 - 1})^{x/b} + (z_1 - \sqrt{z_1^2 - 1})^{x/b}] - \frac{z_1}{z_1^2 - 1} [(z_1 + \sqrt{z_1^2 - 1})^{x/b} - (z_1 - \sqrt{z_1^2 - 1})^{x/b}]] + t_1 [\frac{b - x}{b} [(z_1 + \sqrt{z_1^2 - 1})^{(b-x)/b} + (z_1 - \sqrt{z_1^2 - 1})^{(b-x)/b}] - \frac{z_1}{\sqrt{z^2 - 1}} [(z_1 + \sqrt{z_1^2 - 1})^{(b-x)/b} - (z_1 - \sqrt{z_1^2 - 1})^{(b-x)/b}]] \} .$$

Again, we immediately see that the point $z_1^0 = 1$ is a singular point for our derivative.

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Summary

Si considera il problema di determinare lo spessore e la diffusività di una parete, mediante la cosiddetta matrice di trasmissione, nel caso di flusso termico attraverso una parete le cui temperature al contorno siano funzioni sinusoidali note del tempo, ed i cui flussi corrispondenti siano assegnati. La soluzione completa è data esplicitamente; viene discussa la gamma di validità di tale soluzione, ed esaminata brevemente la sua connessione con una classe di problemi mal posti. Inoltre si studia esplicitamente il legame tra la matrice di trasmissione ed una particolare trasformazione di Lorentz, nell'ambito del calcolo spinoriale.

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