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On the potential distribution in a high vacuum diode (**)

To professor GIANFRANCO CAPRIZ for his sixtieth Birthday

1 - Introduction

The electric potential V in a high vacuum diode is given (see Appendix) by the Child-Langmuir equation [**2**], [**10**]$_1$

$$\Delta V = \frac{K}{V^{\frac{1}{2}}} \qquad K = \frac{J}{\varepsilon}\left(\frac{m}{2q}\right)^{\frac{1}{2}}, \tag{1.1}$$

where m and q are respectively the mass and the electron charge, ε is the permitivity and J the current density.

Let Ω be an open, bounded and doubly-connected subset of R^2 with boundary $\partial\Omega \in C^2$ such that $\partial\Omega = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$. Γ_1 represents the cross-section of the electrons emitting cathode, whereas Γ_2 is the cross-section of the anode. It is assumed that $V = 0$ on Γ_1 and $V = V_P > 0$ on Γ_2. The solution of (1.1) with these boundary conditions gives the potential distribution in the diode.

The purpose of this work is to study the following more general elliptic boundary value problem

$$-\Delta u = \frac{k}{u^{\mu}} \quad \text{in } \Omega, \quad u=0 \quad \text{on } \Gamma_1, \quad u = g \quad \text{on } \Gamma_2, \tag{1.2}$$

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where $\mu > 0$, $k \in R^1$ and g is a function of class $C^2(\bar{\Omega})$ positive on Γ_2. The main feature of interest in (1.2) is that the left hand side of the equation is only defined for $u > 0$, becoming infinite when $u \to 0$. Problems of this type, with singular nonlinearities, have been studied in [**4**], though the results there do not cover completely the present situation.

For a somewhat related work we refer also to [**14**], where a variational approach is used for a problem similar to (1.2).

In this paper we show (Theorem 2.2) that (1.2) cannot have (positive) solutions if $\mu \geqq 1$. The main result is given in Theorem 3.1 where assuming $0 < \mu < 1$, we prove the existence of a negative number $\varkappa$ such that (1.2) is solvable if $k > \varkappa$ and has no solutions when $k < \varkappa$.

2 - Nonexistence of solutions

Let $g_1 = \sup\{g(x), x \in \Gamma_2\} > 0$. On multiplying (1.2) by $1/g_1$, we can always suppose, after redefining k, that $g \leqq 1$.

First of all we state the following easy consequence of the results of G. Giraud [**6**] and G. Prodi [**15**] on the linear Dirichlet problem with left hand side singular on the boundary. We use the notation $d(x) = \text{dist}(x, \Gamma_1)$, $x \in \Omega$, $x = (x_1, x_2)$.

Theorem 2.1. *Let $w(x)$ be given by*

$$\Delta w = 0 \quad \text{in } \Omega, \qquad w = 0 \quad \text{on } \Gamma_1, \qquad w = g \text{ on } \Gamma_2, \tag{2.1}$$

and consider the problem

$$-\Delta v = \frac{k}{w^\mu} \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \Gamma_1, \qquad v = g \text{ on } \Gamma_2. \tag{2.2}$$

If $0 < \mu < 1$ there exists one and only one solution of (2.2) of class $C^{1,1-\mu}(\bar{\Omega})$. If $1 \leqq \mu < 2$ the conclusion continues to hold, except v is in $C^{0,2-\mu}(\bar{\Omega})$.

Proof. By the maximum principle in Hopf's form [**16**] we have $\partial w/\partial \nu > 0$ on Γ_1 (ν is the interior normal). Hence near Γ_1

$$\alpha d(x) \leqq w(x) \leqq \beta d(x) \qquad \beta > \alpha > 0. \tag{2.3}$$

When $0 < \mu < 1$, it follows by a theorem of G. Giraud (see [**6**], p. 50) [1] that the solution of (2.2) exists and belongs to $C^{1,1-\mu}(\bar{\Omega})$. If $2 > \mu \geqq 1$ we can invoke the results of [**15**], which by (2.3) apply directly to (2.2).

(1) Compare also [**12**], p. 100.

Strictly related to the previous theorem is the following

Lemma 2.1. *Let $w(x)$ be as in Theorem* 2.1 *and consider problem* (2.2) *with $k < 0$ and $0 < \mu < 2$. If $1 \leqq \mu < 2$, the solution $v(x)$ of* (2.2) *becomes negative near Γ_1 for every $k < 0$, however small in modulus. If $0 < \mu < 1$ there exists a number $k_1 < 0$ such that $v > 0$ in Ω when $k > k_1$.*

Proof. Let us consider the conformal mapping f which maps $\bar{\Omega}$ one-to-one onto $D = \{X = (X_1, X_2);\ 1 \leqq |X| \leqq e\}$, a mapping which surely exists by virtue of the Riemann theorem. Also, recalling that $\partial\Omega \in C^2$, a result of O. Kellog (see [**11**], p. 116) implies $0 < \lambda \leqq |f'(z)| \leqq \Lambda$. Let $g_1 = \sup\{g(x);\ x \in \Gamma_2\}$ and $g_0 = \inf\{g(x);\ x \in \Gamma_2\}$. Consider the problems

$$(2.4)_i \quad \Delta w_i = 0 \quad \text{in } \Omega\,, \qquad w_i = 0 \quad \text{on } \Gamma_1\,, \qquad w_i = g_i \quad \text{on } \Gamma_2 \qquad (i = 0, 1)\,.$$

By the maximum principle we have $w_0 \leqq w \leqq w_1$. Let

$$(2.5)_i \qquad -\Delta v_i = \frac{k}{w_i^{\mu}} \text{ in } \Omega\,, \qquad v_i = 0 \text{ on } \Gamma_1\,, \qquad v_i = g_i \text{ on } \Gamma_2 \qquad (i = 0, 1)\,.$$

The maximum principle again yields $v_0 \leqq v \leqq v_1$. If we restate $(2.4)_i$ in D, we easily find the corresponding solutions

$$W_i(\varrho, \theta) = g_i \log \varrho\,,$$

where ϱ, θ are polar coordinates in the plane X_1, X_2. Transforming equations (2.2) and $(2.5)_i$ into D they become respectively

$$(2.6) \qquad -\Delta V = \frac{k}{|f'(z)|\, W^{\mu}} \text{ in } D\,, \qquad V = 0 \text{ on } \varrho = 1\,, \qquad V = g \text{ on } \varrho = e\,,$$

$$(2.7)_i \qquad -\Delta V_i = \frac{k}{|f'(z)|\, W^{\mu}} \text{ in } D\,, \qquad V_i = 0 \text{ on } \varrho = 1\,, \qquad V_i = g_i \text{ on } \varrho = e\,.$$

Let $2 > \mu \geqq 1$ and consider

$$(2.8) \qquad -\Delta V^* = \frac{k}{\Lambda g_1 \log \varrho} \quad \text{in } D\,, \qquad V^* = 0 \text{ on } \varrho = 1, \qquad V^* = g_1 \text{ on } \varrho = e.$$

The function V^* supplies an upper barrier for (2.6) i.e. $V^* \geqq V_1 \geqq V$. The solution of (2.8) can be computed explicitly, namely

$$V^*(\varrho, \theta)$$

$$= \frac{N}{2}(\varrho^2 - 1) + \log\varrho \, [g_1 + \frac{N}{2}(1 - e^2) + N \sum_1^\infty \frac{2^n}{n!n} - N(\log\log\varrho + \sum_1^\infty \frac{2^n}{n!n} \log^n \varrho)] \,,$$

where $N = k/g_1 A$. Moreover V^* is negative near $\varrho = 1$ for all $k < 0$, however small in modulus.

Let $1 > \mu > 0$ and consider the lower barrier for (2.6) given by

$$-\Delta V_* = \frac{k}{\lambda g_0 \log^\mu \varrho} \text{ in } D\,, \quad V_* = 0 \text{ on } \varrho = 1\,, \quad V_* = g_0 \text{ on } \varrho = e\,. \tag{2.9}$$

We have

$$V_*(\varrho, \theta) = B \log\varrho - M \sum_1^\infty \frac{2^{n-1}}{(n-1)!(n-\mu)(n+1-\mu)} (\log\varrho)^{n+1-\mu}\,,$$

where $M = \dfrac{k}{\lambda g_0}$ and

$$B = g_0 + M \sum_1^\infty \frac{2^{n-1}}{(n-1)!(n-\mu)(n+1-\mu)}\,.$$

Hence if $0 > k > k_1$ with $k_1 = -\lambda g_0^2 [\sum_1^\infty \dfrac{2^{n-1}}{(n-1)!(n-\mu)(n+1-\mu)}]^{-1}$,

we have $V \geqq V_* > 0$. This completes the proof.

Theorem 2.2. *If $k < 0$ and $\mu \geqq 1$, problem* (1.2) *has no* (*positive*) *solutions.*

Proof. Let $w(x)$ be given by (2.1). If $u(x)$ is a positive solution of (1.2) then, by the maximum principle, $0 < u < w < 1$ in Ω. Consider the problem

$$-\Delta\bar{u} = \frac{k}{w^\beta} \text{ in } \Omega\,, \quad \bar{u} = 0 \text{ on } \Gamma_1\,, \quad \bar{u} = g \text{ on } \Gamma_2\,, \tag{2.10}$$

where $1 \leqq \beta \leqq \mu$ and $\beta < 2$. This problem has a solution by Theorem 2.1, and by Lemma 2.1 it is negative near Γ_1. Moreover $\bar{u} > u$. Hence u is negative somewhere in Ω, which is impossible. This completes the proof.

Theorem 2.3. *Let* $0 < \mu < 1$. *There exists a number* $\bar{k}$ *such that* (1.2) *has no solution when* $k \leqq \bar{k}$.

Proof. Let $w(x)$ be solution of (2.1). Suppose $u(x)$ is a solution of (1.2) with $k < 0$ and consider the problem

$$-\Delta q = k \quad \text{in } \Omega\,, \qquad q = 0 \quad \text{on } \Gamma_1\,, \qquad q = g \quad \text{on } \Gamma_2\,.$$

Since $k/u^\mu < k$ we have $q > u$. Now if $|k|$ is sufficiently large, q assumes negative values in Ω. In fact let φ be given by $-\Delta\varphi = 1$ in Ω, $\varphi = 0$ on $\partial\Omega$. Let $\varphi(\bar{x}) = \sup\{\varphi(x);\ x \in \Omega\}$, $\bar{x} \in \Omega$. We have $q(\bar{x}) = w(\bar{x}) + k\varphi(\bar{x}) < 1 + k\varphi(\bar{x})$. Hence if $k < -1/\varphi(\bar{x})$ problem (1.2) has no solution.

When $k > 0$ we can give the following result of existence and uniqueness. For the idea of the proof see Theorem 1.1 in [**4**].

Theorem 2.4. *If* $k > 0$ *and* $0 < \mu < \infty$ *there exists one and only one solution to problem* (1.2).

Proof. Let u_1, u_2 be solutions to (1.2). Put $\hat{u} = u_1 - u_2$ and $A = \{x \in \Omega;\ \hat{u}(x) < 0\}$. Since in A $k/u_2^\mu < k/u_1^\mu$, we have $-\Delta\hat{u} > 0$. Moreover $\hat{u} = 0$ on ∂A hence $\hat{u} > 0$ in A so that $A = \emptyset$. It follows $u_1 \geqq u_2$ in Ω. Similarly we can prove $u_1 \leqq u_2$. Hence $u_1 = u_2$.

$$(2.11) \qquad -\Delta u_\varepsilon = \frac{k}{(u_\varepsilon + \varepsilon)^\mu} \quad \text{in } \Omega\,, \quad u_\varepsilon = 0 \quad \text{on } \Gamma_1\,, \quad u_\varepsilon = g \quad \text{on } \Gamma_2\,.$$

For all $\varepsilon > 0$, $u_- = w$ given by (2.1) is a subsolution for (2.11). A supersolution u_+ is obtained as solution of the problem

$$-\Delta u_+ = \frac{k}{\varepsilon^\mu} \quad \text{in } \Omega\,, \quad u_+ = 0 \quad \text{on } \Gamma_1\,, \quad u_+ = g \quad \text{on } \Gamma_2\,.$$

Since $u_+ > u_-$, (2.11) has a solution by a theorem of Amann [**1**]. Uniqueness for (2.11) can be obtained exactly as before.

Let $0 < \varepsilon < \delta$. We prove that (i) $u_\varepsilon > u_\delta$ and (ii) $\varepsilon + u_\varepsilon \leqq \delta + u_\delta$. Put $u' = u_\varepsilon - u_\delta$ and $B = \{x \in \Omega;\ u'(x) < 0\}$. In B we have $k/(u_\varepsilon + \varepsilon)^\mu > k/(u_\delta + \delta)^\mu$ so that $-\Delta u' \geqq 0$ and by the maximum principle $u' > 0$. Hence $B = \emptyset$ which proves (i). Let now $\tilde{u} = (u_\delta + \delta) - (u_\varepsilon + \varepsilon)$ and $C = \{x \in \Omega;\ \tilde{u}(x) < 0\}$. Since $-\Delta\tilde{u} = k/(u_\delta + \delta)^\mu - k/(u_\varepsilon + \varepsilon)^\mu > 0$ in C and $\tilde{u} = 0$ on ∂C we get $\tilde{u} > 0$

in C. This proves (ii). It follows that $u_\delta(x)$ converges uniformly to a function $u(x) \in C^0(\bar{\Omega})$ such that $u = 0$ on Γ_1 and $u = g$ on Γ_2. To see that $-\Delta u = k/u^\mu$ and $u \in C^2(\Omega)$ we can proceed as in [4].

Remark. Suppose the assumptions of Theorem 2.4 hold true. Let $0 < \mu < 1$ and consider the problem

$$-\Delta \bar{u} = \frac{k}{w^\mu} \quad \text{in } \Omega, \qquad \bar{u} = 0 \quad \text{on } \Gamma_1, \qquad \bar{u} = g \quad \text{on } \Gamma_2,$$

where w is given by (2.1). By the maximum principle we have $w < u < \bar{u}$ and by Theorem 2.1 $\bar{u} \in C^{1,1-\mu}(\bar{\Omega})$. Hence the interior normal derivative on Γ_1 of $\bar{u}$ exists and satisfies $0 < \partial w/\partial \nu < \partial \bar{u}/\partial \nu$. Therefore near Γ_1 we have $\alpha d(x) < u < \beta d(x)$, $\beta > \alpha > 0$. Again by Theorem 2.1 we get $u \in C^{1,1-\mu}(\bar{\Omega})$.

When $\mu \geqq 1$ we can apply with minor changes the results of Theorems 2.2 and 2.5 of [4] obtaining $u \in C^{0,2/(1+\mu)}(\bar{\Omega})$.

3 - The case $k < 0$

In this Section we treat the case of physical relevance, $k < 0$. Our main result is the following

Theorem 3.1. *There exists a negative number $\varkappa$ such that:*

(i) *if $\varkappa < k \leqq 0$ equation* (1.2) *has at least one solution;*
(ii) *if $-\infty < k < \varkappa$ problem* (1.2) *in not solvable.*

The proof of Theorem 3.1 is based on the following lemmas.

Lemma 3.1. *Let us consider the following one dimensional version of problem* (1.2)

$$-z''(s) = \frac{l}{(z(s))^\mu}, \qquad z(0) = 0, \qquad z(L) = \beta, \tag{3.1}$$

where $0 < \mu < 1$, $L > 0$, $\beta > 0$ and $l < 0$. If $\beta < \bar{\beta}$, where

$$\bar{\beta} = \left[\frac{\sqrt{aL}(1+\mu)}{2}\right]^{2/(1+\mu)} \qquad \text{and} \qquad a = \frac{2l}{\mu - 1}, \tag{3.2}$$

problem (2.1) *has no solution. If $\beta \geqq \bar{\beta}$ then* (3.1) *has a unique solution, which* *when* $\beta = \bar{\beta}$, *is given by*

$$z(s) = [\frac{\sqrt{a}}{2}(1+\mu)s]^{2/(1+\mu)}.$$

Proof. Since $z > 0$ in $(0, L)$, we have $z'' > 0$. It follows $z'(s) > 0$, $s \in (0, L)$ and

$$z' = (az^{1-\mu} + C)^{\frac{1}{2}}, \tag{3.3}$$

where C is a constant of integration certainly nonnegative. Integrating (3.3) by separation of variables, condition $z(L) = \beta$ becomes

$$\Phi(C) = L \qquad \text{with} \qquad \Phi(C) = \int_0^\beta \frac{dz}{(az^{1-\mu} + C)^{\frac{1}{2}}}. \tag{3.4}$$

It is easy to verify that (3.4) has one and only one solution if $\beta \leqq \bar{\beta}$ and no solution when $\beta > \bar{\beta}$.

Lemma 3.2. *Let* $0 < L_1 < L_2$ *and* w_i *be defined by*

$$\Delta w_i = 0 \quad \text{in } \Omega, \qquad w_i = 0 \quad \text{on } \Gamma_1, \qquad w_i = L_i \quad \text{on } \Gamma_2 \qquad (i = 1, 2).$$

Then

$$0 < \frac{\partial w_1}{\partial \nu} < \frac{\partial w_2}{\partial \nu}, \tag{3.5}$$

where ν *is the interior normal on* Γ_1.

Proof. By the maximum principle $w_2 > w_1$ and since $w_1 = w_2$ on Γ_1, we get (3.5).

Lemma 3.3. *Let* $0 < \mu < 1$ *and* $k < 0$. *If* $|k|$ *is sufficiently small there exists a weak subsolution for* (1.2) *i.e. a function* u_- *which satisfies*

$$u_- \in H^1(\Omega)\ (^2) \qquad u_- - g \in H_0^1(\Omega), \qquad u_- > 0 \quad \text{in } \Omega, \tag{3.6}$$

$$\frac{1}{u_-^\mu} \in L^p(\Omega) \quad \textit{for a certain } p > 1, \tag{3.7}$$

$$\int_\Omega \nabla u_- \cdot \nabla v \, dx \leqq \int_\Omega k \frac{v}{u_-^\mu} dx \qquad \textit{for all } v \in H_0^1(\Omega) \qquad v \geqq 0. \tag{3.8}$$

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(2) H^1, H_0^1 are the usual Sobolev spaces, see e.g. [7].

Proof. Let $w(x)$ be given by (2.1). By the maximum principle in Hopf's form we have $\partial w/0\nu > 0$ on Γ_1. Suppose $0 < L < g_0$ and define $\hat{\Omega}=\{x \in \Omega;\ 0 < w(x) < L\}$, $\Gamma_3 = \partial\hat{\Omega} - \Gamma_1$. Clearly $\Gamma_1 \cap \Gamma_2 = \emptyset$, and $w = L$ on Γ_3. Choose L so small that

$$|\nabla w| \geqq m > 0 \qquad \text{in } \hat{\Omega}\,. \tag{3.9}$$

By the implicit function theorem, Γ_3 which is defined by $w(x_1, x_2) = L$, is a C^1 curve. Let us consider problem (3.1) with L as above. Take l and β in Lemma 3.1 such that $\beta > \bar{\beta}$ and put $\varphi(x) = z(w(x))$. For every β and l $(\beta > \bar{\beta})$, $z(s)$ tends linearly to zero as $s \to 0$. Hence $1/\varphi^\mu \in L^p(\hat{\Omega})$ with $p > 1$.

By (3.1) and (3.9) we have in $\hat{\Omega}$

$$-\Delta\varphi = -z''|\nabla w|^2 - z'\Delta w \leqq \frac{k}{\varphi^\mu} \qquad k = lm\,. \tag{3.10}$$

Define $\Omega^* = \Omega - \hat{\Omega}$ and consider the problem

$$\Delta\Psi = 0 \quad \text{in } \Omega^*\,, \qquad \Psi = \beta \quad \text{on } \Gamma_3\,, \qquad \Psi = g \quad \text{on } \Gamma_2\,.$$

Let n be the interior unit normal to Γ_3 with respect to Ω^*. We have

$$\frac{\partial\varphi}{\partial n} = z'(L)\,\frac{\partial w}{\partial n} \qquad \text{on } \Gamma_3\,. \tag{3.11}$$

Now if β (and correspondingly l) are taken sufficiently small, $z'(L)$ and by (3.11), $\partial\varphi/\partial n$ can be made arbitrarily small. Since $\partial\Psi/\partial n > 0$ on Γ_3 we can choose β and l so that

$$\frac{\partial\varphi}{\partial n} < \frac{\partial\Psi}{\partial n} \qquad \text{on } \Gamma_3\,. \tag{3.12}$$

Take now the function Θ defined by

$$-\Delta\Theta = -\varepsilon^2 \quad \text{in } \Omega^*\,, \qquad \Theta = \partial\Omega^*\,, \tag{3.13}$$

and put $\Phi = \Psi + \Theta$. Since $|\nabla\Theta|$ tends uniformly to zero as $\varepsilon \to 0$, recalling (3.12) we have

$$\frac{\partial\Phi}{\partial n} > \frac{\partial\varphi}{\partial n} \qquad \text{on } \Gamma_3\,, \tag{3.14}$$

and $\Phi > \lambda > 0$ in Ω^*, when ε is sufficiently small. Hence taking $|k|$ still smaller if necessary, we can write

$$-\varepsilon^2 = -\Delta\Phi \leqq \frac{k}{\Phi^\mu} \quad \text{in } \Omega^* . \tag{3.15}$$

Let $v \in H_0^1(\Omega)$, $v \geqq 0$. By (3.15) we have

$$\int_{\Omega^*} \nabla\Phi\cdot\nabla v \,dx + \int_{\Gamma_3} v \frac{\partial\Phi}{\partial n}\,ds \leqq k\int_{\Omega^*} \frac{v}{\Phi^\mu}\,dx , \tag{3.16}$$

and by (3.10)

$$\int_{\hat\Omega} \nabla\varphi\cdot\nabla v \,dx - \int_{\Gamma_3} v \frac{\partial\varphi}{\partial n}\,ds \leqq \int_{\hat\Omega} \frac{v}{\varphi^\mu}\,dx . \tag{3.17}$$

The integral on the right hand side of (3.17) exists since $\varphi^{-\mu} \in L^p(\hat\Omega)$, $p > 1$ and $v \in L^q(\Omega)$, $q < \infty$ by the Sobolev's inbedding theorem. Put

$$u_- = \begin{cases} \varphi(x) & x \in \hat\Omega , \\ \Phi(x) & x \in \Omega^* . \end{cases}$$

Adding up (3.16) and (3.17) and recalling (3.14) we obtain (3.8).

P r o o f o f T h e o r e m 3.1. If $|k|$ is sufficiently small, say $k = k_0$, by Lemma 3.3 there exists a weak subsolution for problem $(1.2)_{k_0}$.

Let us consider the sequence of linear problem

$$u_m - g \in H_0^1(\Omega) , \quad \int_\Omega \nabla u_m\cdot\nabla v \,dx = k_0 \int_\Omega \frac{v}{u_{m-1}^\mu}\,dx \qquad \text{for all } v \in H_0^1(\Omega) , \tag{3.18}$$

where w is given by (2.1). The sequence $\{u_m\}$ is well defined in $H^1(\Omega)$ and $u_m > u_-$ for all m. In fact we have $u_0 > u_-$ in Ω. Moreover suppose $u_{m-1} - g \in H_0^1$ and $u_{m-1} > u_-$. Subtracting (3.8) from (3.18) we get

$$\int_\Omega \nabla(u_m - u_-)\cdot\nabla v \,dx \geqq 0 \qquad \text{for all } v \in H_0^1(\bar\Omega) \quad v \geqq 0 .$$

Hence by the weak maximum principle ([7], p. 39) we have $u_m > u_-$.

Again by induction we can prove that $\{u_m\}$ is nonincreasing. Putting $v = u_m - w$ in (3.18), we obtain

$$\int_\Omega |\nabla u_m|^2 dx \leqq \|\nabla u_m\|_{L^2} \times \|\nabla w\|_{L^2} + k_0 \|w/u_-^\mu\|_{L^1} .$$

It follows that $\{u_m\}$ is «a priori» bounded in the H^1-norm. Hence we can extract a subsequence still denoted u_m, weakly convergent in H^1 to a function $u \in H^1$ such that $u - g \in H^1_0$. Moreover u_m converges in L^2 to u. Thus

$$\lim_{m\to\infty} u_m(x) = \inf\{u_m(x);\, m \in N\} = u(x)\,.$$

Pick $v \in H^1_0(\Omega)$ and let $v = v^+ - v^-$, $v^+ = \sup\{v, 0\}$, $v^- = \sup\{-v, 0\}$. The sequences $\{u^\pm/u^\mu_m\}$ are nondecreasing. Moreover

$$\lim_{m\to\infty} \frac{v^\pm}{u^\mu_m} = \frac{v^\pm}{u^\mu} \in L^1(\Omega)\,.$$

Hence by B. Levi's Theorem

$$\lim_{m\to\infty} \int_\Omega \frac{v^\pm}{u^\mu}\, dx = \int_\Omega \frac{v^\pm}{u^\mu}\, dx\,. \tag{3.19}$$

Passing to the limit in (3.18), we obtain

$$\int_\Omega \nabla u\cdot\nabla v\, dx = k_0 \int_\Omega \frac{v}{u^\mu}\, dx \qquad \text{for all } v \in H^1_0(\Omega)\,.$$

Since $k_0/u^\mu \in L^p(\Omega)$, $p > 1$, we have [**9**] $u \in C^0(\bar\Omega) \cap C^\infty(\Omega)$. On the other hand $w \geqq u \geqq u_-$. Hence near Γ_1 we have $\alpha d(x) \geqq u(x) \geqq \beta d(x)$ with $\alpha > \beta > 0$. Thus by Theorem 2.1 we obtain $u \in C^{1,1-\mu}(\bar\Omega)$.

Let now $\varkappa$ be the infimum of all k such that one can solve $(1.2)_k$. By Theorem 2.3 $0 > k_0 \geqq \varkappa > -\infty$. Suppose $\varkappa < k \leqq 0$. Clearly there exists $k^* < k$ such that $(1.2)_{k^*}$ has a solution u_* and $-\Delta u_* = k^*/u^\mu_* \leqq k/u^\mu_*$. Hence u^* is a subsolution for problem $(1.2)_k$. Since w given by (2.1) is a supersolution, we conclude that $(1.2)_k$ is solvable. This completes the proof.

Remark 1. One question arises naturally: why problem (1.2) may not have solutions when a potential distribution exists with the same values of parameters? To answer it is necessary to remind that equation (1.2), although sufficient for many practical purposes [**5**], gives a rather rough picture of what really happens. Various other aspects (in particular the temperature) should be kept into account to get a more accurate model. We refer to [**13**] for a complete discussion of physical background and history of this problem.

Remark 2. In [**14**] it is proved a result closely related to part (i) of Theorem 3.1. This is done (see first part of Theorem II in [**14**]), when Ω is a disk of R^n, but the argument can be extended to the case of an arbitrary connected domain.

The technique of the proof used here for a doubly connected domain is however completely different.

Appendix

A diode is an electron tube consisting of a heated cathode which operates as an emitter of electrons and an anode (or plate), serving as a collector. Both electrodes are placed in an evacuated glass envelope.

When the anode is made positive with respect to the cathode, electrons flow between the electrodes and through an external circuit. The cloud of electrons leaving the cathode influences the electric potential inside the tube which is given, using the notations of the Introduction by

$$\Delta V = -\varrho/\varepsilon . \tag{a}$$

To express the charge density ϱ in terms of V we equate the kinetic energy acquired by the electron as it moves through the field, to the potential energy change i.e.

$$Vq = mu^2/2 , \tag{b}$$

where u is the velocity of the electron when the potential is V. We assume $u = 0$ if $V = 0$. Moreover we have

$$J = -\varrho u , \tag{c}$$

which is a sort of continuity equation. Eliminating ϱ and u in (a), (b) and (c) we get equation (1.1).

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Sommario

In questo lavoro si studia un problema al contorno ellittico nonlineare per l'equazione di Child-Langmuir che fornisce il potenziale in un diodo ad alto vuoto. Vengono dati vari risultati di esistenza e unicità della soluzione.

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