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GIUSEPPE ROSOLINI (*)

# Domains and dominical categories (**)

Dominical categories, first introduced by A. Heller in [2], are an axiomatic approach to categories of partial morphisms. They appear to be an appropriate abstract tool to study recursion, cf. [1], [3]. We refer the reader to [2] for further discussions on dominical categories.

The aim of the first three sections is to show how the axiomatic approach fits the job: every dominical category is a category of partial maps of a suitable category. In the last section we try to convince the reader that a dominical category with a Turing morphism $u: X \times X \to X$ can be thought of as a category of sets and partially defined maps with an enumeration from $X$ onto the set of partial maps from $X$ into itself, in a suitably weak theory of sets.

We introduce definitions and notions. Then we build the category of domains, and show that its category of partial maps nicely embeds the given one. In order to do this we discuss a few topics about categories of partial maps. Then we shall produce a topos, into which the dominical category embeds preserving all the structure of dominical category.

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## 1 - Dominical categories

As said in the introduction, the notion of dominical category is intended to axiomatize categories of partial maps.

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(*) Indirizzo: Dipartimento di Matematica, Via Università 12, 43100 Parma, Italy.
(**) Ricevuto: 4-VII-1984.

We say that a category $C$ is *pointed* if for every pair of objects $X$ and $Y$ there is a map $0: X \to Y$ such that composition with any map gives the corresponding map 0. In a pointed category we call a map $f$ total if it reveals zero maps: if $f \circ w = 0$, then $w = 0$.

Any identity map is total and composition of total maps is total: the subcategory of $\boldsymbol{C}$ whose maps are the total morphisms is denoted by $\boldsymbol{C}_t$.

Def. 1.1. A category $\boldsymbol{C}$ is *dominical* if it is pointed and is endowed with a functor $- \times + : C \times \boldsymbol{C} \to \boldsymbol{C}$ of pointed categories which will be called *product* such that, when restricted to the category $\boldsymbol{C}_t$ of total maps, it is an actual categorical product. Moreover, projections and diagonals, which are defined in the category $\boldsymbol{C}_t$, satisfy the following conditions: for any map $w: X \to Z$ in $\boldsymbol{C}$

$$w \times 0 = 0 \times w = 0 , \quad p(w \times \mathrm{id}) = w \circ p ,$$

$$q(\mathrm{id} \times w) = w \circ q , \quad (w \times w)\, d = d \circ w ,$$

where $p$, $q$ and $d$ are the first and second projections and the diagonal map respectively. Finally, the isomorphisms $a$ and $t$ for associativity and commutativity defined, as usual, by

$$a = \langle\langle p, p \circ q\rangle, q \circ q\rangle \quad \text{and} \quad t = \langle q, p\rangle ,$$

are natural on $\boldsymbol{C}$ in all variables.

Examples 1.2. We first quote the obvious ones: the category of sets and partial maps; the category of topological spaces and maps defined on open subsets; the groupoid of partial recursive functions of natural numbers. Another non-trivial example is the category of posets with a bottom element and maps preserving order and bottom.

Def. 1.3. A functor $F: \boldsymbol{C} \to \boldsymbol{D}$ between dominical categories is *dominical* if it preserves zero maps and products.

We must now recall a notion which Heller introduces in [**2**]. Given a map $w: X \to Y$ in the dominical category $\boldsymbol{C}$ we write $\mathrm{dom}\, w$ for the map $p(\mathrm{id} \times w)\, d$: $X \to X$, where $p$ is the first projection from $X \times Y$ and $d$ is the diagonal map of $X$ into $X \times X$. Below we list a few properties of domain maps.

Proposition 1.4. *The domain operator has the following properties*:

(i) $\mathrm{dom}(\mathrm{dom}\, w) = \mathrm{dom}\, w$;

(ii) $\mathrm{dom}\, w \circ \mathrm{dom}\, z = \mathrm{dom}\, z \circ \mathrm{dom}\, w = \mathrm{dom}(\mathrm{dom}\, w \circ \mathrm{dom}\, z)$;

(iii) $\operatorname{dom} w \circ \operatorname{dom} w = \operatorname{dom} w$;
(iv) $\operatorname{dom}(w \circ z) = \operatorname{dom}((\operatorname{dom} w) \circ z)$;
(v) $w = w \circ \operatorname{dom} w$;
(vi) $\operatorname{dom} f = \mathrm{id}$ *if and only if* $f$ *is total*;
(vii) $\operatorname{dom}(w \times z) = \operatorname{dom} w \times \operatorname{dom} z$.

Proof. See [2] for (i)-(iv). To prove (v), just recall the definition of dominical category to check that $w = p \circ d \circ w = p(w \times w)d = p(w \times \mathrm{id})(\mathrm{id} \times w)d = w \circ p(\mathrm{id} \times w)d = w \circ \operatorname{dom} w$. One implication in (vi) follows from the fact that product is categorical product on $C_t$. The other one follows from (iv) and (v). As to (vii), notice that the map

$$\langle p \times p, q \times q \rangle : (X \times Y) \times (Z \times W) \to (X \times Z) \times (Y \times W)$$

can be obtained as composition of isomorphisms for associativity and commutativity, hence it is natural in all variables; thus evaluate

$$\operatorname{dom}(w \times z) = p((\mathrm{id} \times \mathrm{id}) \times (w \times z))\, d = (p \times p)((\mathrm{id} \times w) \times (\mathrm{id} \times z))(d \times d)$$

$$= (p(\mathrm{id} \times w)d) \times (p(\mathrm{id} \times z)d) = \operatorname{dom} w \times \operatorname{dom} z$$

— beware: we forgot about the indices!

## 2 - Categories of partial maps

The main examples of dominical categories are categories of partial maps. We treat them rather extensively in this section as we do not have a good reference for the subject.

Let $A$ be a category and let $M$ be a family of monics of $A$ closed under identities and composition with the property that any pullback of a monic in $M$ exist in $A$ and a representative of it belongs to $M$. We shall call such a family a *notion of partial*. We must point out that the last request on $M$ does not yield that $M$ is closed under isomorphic monics.

Def. 2.1. Given a notion of partial $M$ in $A$, a partial map $(m: A \rightarrowtail X, w: A \to Y)$ in $A$ said to be *defined in* $M$, if $m$ is in $M$. Two partial maps $(m: A \rightarrowtail X, w: A \to Y)$ and $(n: B \rightarrowtail X, z: B \to Y)$ are *equivalent* if there is an isomorphism $i: A \to B$ making the two obvious diagrams commute.

The conditions on $M$ are forced to be like that as soon as one tries to define a category of partial maps: the category $P(A/M)$ of partial maps defined on $M$ has the same objects as $A$ and equivalence classes of partial maps defined on monics in $M$ as arrows. Composition of two maps whose representatives are $(m: A \rightarrowtail X,\ w: A \to Y)$ and $(n: B \rightarrowtail Y,\ z: B \to Z)$ is just represented by $\big(m\circ w^{-1}(n): w^{-1}(B) \rightarrowtail X,\ z\circ n^{*}(w): w^{-1}(B) \rightarrowtail Z\big)$, where $w^{-1}(B)$ and the maps from it form the pullback of $n$ and $w$, as usual.

Suppose $A$ has a strict initial object; we say that the notion of partial $M$ is *decent* if for any monic $m$ which is not iso, there is a map which pulls it back to the least subobject.

We can leave the proof of the following statement to the reader.

Proposition 2.2. *Let $A$ be a category with binary products and a strict initial object, let $M$ be a decent notion of partial. If all monics of the form $0 \rightarrowtail X$ are in $M$, then $P(A/M)$ is a dominical category.*

There is a simple way to describe domains in the dominical sense in $P(A/M)$.

Lemma 2.3. *In the assumptions of 2.2, in $P(A/M)$ the domain of a map $(m: A \rightarrowtail X,\ w: A \to Y)$ is the equivalence class determined by the pair $(m: A \rightarrowtail X,$ $m: A \rightarrowtail X)$.*

Proof. Recall that the composition defining a domain is built with first projection and diagonal which are total maps. So we just need to check what the composition

$$(\mathrm{id}\times m: X\times A \rightarrowtail X\times X,\ p: X\times A \to X)\circ(\mathrm{id}: X \rightarrowtail X,\ d: X \to X\times X)$$

gives. As a pullback of $\mathrm{id}\times m$ along $d$ is $m$ itself, which belongs to $M$, the result follows immediately.

## 3 - The category of domains

We intend to produce a category for which the dominical category $C$ constitutes a category of partial arrows. Think of the domains as a sort of restriction maps: we shall build a category on these maps as objects.

Let Dom($C$) denote the « category of domains of $C$ ». Its objects are the maps in $C$ of the form $\mathrm{dom}\, w$ and its arrows $f: \mathrm{dom}\, w \to \mathrm{dom}\, z$ are the maps in $C$ which are defined on $\mathrm{dom}\, w$ and take values in $\mathrm{dom}\, z$, in the sense that

$$\mathrm{dom}\, f = \mathrm{dom}\, w \quad \text{and} \quad f = \mathrm{dom}\, z\circ f\,.$$

Identity maps are the domains themselves and composition of $f$ and $g$ in $\mathrm{Dom}(\boldsymbol{C})$ is $g\circ f$ as composed in $\boldsymbol{C}$; indeed, if $f\colon \mathrm{dom}\, w \to \mathrm{dom}\, z$ and $g\colon \mathrm{dom}\, z \to \mathrm{dom}\, v$, then $\mathrm{dom}(g\circ f) = \mathrm{dom}((\mathrm{dom}\, g)\circ f) = \mathrm{dom}((\mathrm{dom}\, z)\circ f) = \mathrm{dom}\, f = \mathrm{dom}\, w$, and $g\circ f = \mathrm{dom}\, v\circ g\circ f$.

The category $\boldsymbol{C}_t$ of total morphisms in $\boldsymbol{C}$ is embedded into $\mathrm{Dom}(\boldsymbol{C})$ by taking an object to its identity map and a map in $\boldsymbol{C}$ to itself. We shall show that the category $\mathrm{Dom}(\boldsymbol{C})$ is the natural completion of $\boldsymbol{C}$ with respect to domains. In order to do this we need the following

Lemma 3.1. (i) *Let* $w\colon X \to Y$ *be a map in* $\boldsymbol{C}$; *then* $w$ *induces a map* $w\colon \mathrm{dom}\, w \to \mathrm{id}_Y$ *in* $\mathrm{Dom}(\boldsymbol{C})$. (ii) *Let* $v\colon \mathrm{dom}\, w \to \mathrm{dom}\, z$; *then* $v = \mathrm{dom}\, z\circ v\circ \mathrm{dom}\, w$. (iii) *Let* $\mathrm{dom}\, v\colon \mathrm{dom}\, w \to \mathrm{dom}\, z$; *then* $\mathrm{dom}\, v = \mathrm{dom}\, w$ *and it is a monic in* $\mathrm{Dom}(\boldsymbol{C})$. (iv) *Let* $w, z\colon Z \to X\times Y$ *be maps in* $\boldsymbol{C}$ *and suppose* $pw = pz$ *and* $qw = qz$; *then* $w = z$.

Proof. (i) Follows from 1.4(v). (ii) As $\mathrm{dom}\, v = \mathrm{dom}\, w$, $v = v\circ \mathrm{dom}\, w$ and the result follows from the very definition of map in $\mathrm{Dom}(\boldsymbol{C})$. (iii) By definition, $\mathrm{dom}\, w = \mathrm{dom}(\mathrm{dom}\, v) = \mathrm{dom}\, v$. Hence, for any $y$ with target $\mathrm{dom}\, w$ in $\mathrm{Dom}(\boldsymbol{C})$, $\mathrm{dom}\, v\circ y = f$: this yields the assertion. (iv) The map $\langle p, q\rangle\colon X\times Y \to X\times Y$ is the identity and is equal to $(p\times q)d$. Now consider the following equalities: $w = (p\times q)dw = (p\times q)(w\times w)d = (pw\times qw)d = (pz\times qz)d = (p\times q)(z\times z)d = z$.

We can now give a more detailed picture of $\boldsymbol{C}_t$ in $\mathrm{Dom}(\boldsymbol{C})$.

Theorem 3.2. *The embedding of* $\boldsymbol{C}_t$ *in* $\mathrm{Dom}(\boldsymbol{C})$ *is full and faithful, and preserves products.*

Proof. Full-and-faithfulness is trivial. As to product-preserving, take $X\times Y$ in $\boldsymbol{C}_t$ and suppose $f\colon \mathrm{dom}\, w \to \mathrm{id}_X$ and $g\colon \mathrm{dom}\, w \to \mathrm{id}_Y$. Then we want to prove that the map $h = (f\times g)\circ d$ is the unique map from $\mathrm{dom}\, w$ into $\mathrm{id}_X\times \mathrm{id}_Y$ induced by $f$ and $g$ which does the right job. To show that $h$ is defined on $\mathrm{dom}\, w$:

$$\mathrm{dom}\, h = \mathrm{dom}((f\times g)\circ d) = \mathrm{dom}(\mathrm{dom}(f\times g)\circ d) = \mathrm{dom}((\mathrm{dom}\, w\times \mathrm{dom}\, w)\circ d)$$

$$= \mathrm{dom}(d\circ \mathrm{dom}\, w) = \mathrm{dom}(\mathrm{dom}\, d\circ \mathrm{dom}\, w) = \mathrm{dom}(id\circ \mathrm{dom}\, w) = \mathrm{dom}\, w\,.$$

It is easy to check that composition of $h$ with projections give either component. Finally, 3.1(iv) give uniqueness of $h$.

We shall identify $C_t$ with its image in Dom($C$), and call its objects *total.* Now we move on to the study of Dom($C$).

Theorem 3.3. *The category* Dom($C$) *has binary products and a strict initial object; it has inverse images of monics of the form* dom $w$. *Thus the family* $D$ *of domains is a decent notion of partial. The category* $C$ *is embedded into the category* $P(\mathrm{Dom}(C)/D)$ *of partial maps defined on domains via a dominical functor* $I$.

Proof. The product $\mathrm{dom}\, w \times \mathrm{dom}\, z$ is just $\mathrm{dom}(w \times z)$: the proof of this follows from the fact that it embeds naturally into $X \times Y$, if these are the total objects on which $w$ and $z$ are given. We prove that the domain dom 0 of any zero map in $C$ is strictly initial. Let dom $w$ be any object of Dom($C$); then the appropriate map $0: \mathrm{dom}\, 0 \to \mathrm{dom}\, w$ is the only one connecting the two since any other $f$ must be equal to $f \circ \mathrm{dom}\, 0 = 0$. As any $g$ into dom 0 must equal $\mathrm{dom}\, 0 \circ g = 0$, the initial object is strict. The second part is a restatement of a property given in [2], the inverse image of dom $w$ along $f$ being $\mathrm{dom}(\mathrm{dom}\, w \circ f)$. The embedding $I$ is defined by taking a map $w: X \to Y$ in $C$ to the partial map between total domains $X$ and $Y$ in Dom($C$) defined on the subobject dom $w$ of $X$ by $w$ itself. We leave to the reader to check that everything works fine. Notice that a zero map goes to the partial map defined on the least subobject.

Without any other formal statement we think that this already gives a clear picture of Dom($C$); any map in it represents the partial map induced by a map in $C$ possibly corestricted to a domain, as long as this corestriction does not influence it.

Corollary 3.4. *The image of the embedding of* $C$ *into the category* $P(\mathrm{Dom}(C)/D)$ *is the full subcategory of the total objects.*

## 4 - Representation of dominical categories

In this section we first treat the problem of « good choice » of representatives for subobjects and show how generic the embedding of 3.3 is.

Let $A$ be a category and let $M$ be a decent notion of partial. After forming $P(A/M)$ one is led to consider $\mathrm{Dom}(P(A/M))$, hoping to get $A$ back. The aim can be achieved successfully.

Theorem 4.1. *Let* $A$ *and* $M$ *be as above. Then there is a functor* $J: A \to \mathrm{Dom}(P(A/M))$ *which is full, faithful and representative.*

Proof. For $X$ an object of $\boldsymbol{A}$, define $J(X)$ as the equivalence class represented by the pair $(\mathrm{id}\colon X \to X, \mathrm{id}\colon X \to X)$, which is a domain by 2.3. For $f\colon X \to Y$ in $\boldsymbol{A}$, let $J(f)$ be the «partial map» represented by $(\mathrm{id}\colon X \to X, f\colon X \to Y)$. As to representativity, recall from 2.3 that an object of $\mathrm{Dom}(P(\boldsymbol{A}/M))$ is represented by a pair $(m\colon A \rightarrowtail X, m\colon A \rightarrowtail X)$, with $m$ in $M$. This object is isomorphic to $J(A)$.

To have a picture of what $\mathrm{Dom}(P(\boldsymbol{A}/M))$ is, think of its objects as equivalence classes of «superobjects» of $\boldsymbol{A}$, see [4]. In order to fix this idea, we introduce the following notion.

Def. 4.2. The functor $J$ appearing in the conclusion of 4.1 being an equivalence, an inverse to $J$ is called a *good choice* for representatives for the family $M$.

Examples 4.3. In the category of sets, with $M$ the class of all monics, images are a good choice for representatives.

As an argument in favour of the definition given in 4.2, we state a simple criterion for good choices.

Proposition 4.4. *Let $\boldsymbol{A}$ and $M$ be as in* 4.1. *If any two monics $m$ and $n$ in $M$ which are equal up to an isomorphism $i$ (say, $m = n \circ i$) are actually equal, then $M$ has a good choice for representatives.*

Indeed, the construction of taking the category of domains, will lead us to a complete family of examples.

Theorem 4.5. *Let $C$ be a dominical category and let* $\mathrm{Dom}(\boldsymbol{C})$ *be the category of its domains. Then there is a good choice for representatives for the family $D$ of domain maps.*

Proof. Define the functor $K\colon \mathrm{Dom}(P(\mathrm{Dom}(\boldsymbol{C})/D)) \to \mathrm{Dom}(\boldsymbol{C})$ as

$$K(\mathrm{dom}\, w\colon \mathrm{dom}\, w \to \mathrm{dom}\, z,\ \mathrm{dom}\, w\colon \mathrm{dom}\, w \to \mathrm{dom}\, z) = \mathrm{dom}\, w\,.$$

It is an application of 4.4; if $\mathrm{dom}\, w\colon \mathrm{dom}\, w \to \mathrm{dom}\, z$ is isomorphic to $\mathrm{dom}\, v\colon \mathrm{dom}\, v \to \mathrm{dom}\, z$ via $i$, then $\mathrm{dom}\, w = \mathrm{dom}\, v \circ i = \mathrm{dom}\, v \circ \mathrm{dom}\, w = \mathrm{dom}\, w \circ \mathrm{dom}\, v = \mathrm{dom}\, v$. Thus $K$ is well-defined on objects. Definition of $K$ on maps is obvious. It is straightforward to prove that $J$ and $K$ form an equivalence of categories.

We are now in a position to give the property which characterizes $\mathrm{Dom}(\boldsymbol{C})$.

Theorem 4.6. *Let $\boldsymbol{C}$ be a dominical category, let $I$ be the embedding into $P(\mathrm{Dom}(\boldsymbol{C})/D)$. Let $\boldsymbol{A}$ be a category with binary products and a strict initial object, let $M$ be a decent notion of partial with a good choice for representatives. Given any dominical functor $F\colon \boldsymbol{C} \to P(\boldsymbol{A}/M)$, there is a unique functor $F'\colon \mathrm{Dom}(\boldsymbol{C}) \to \boldsymbol{A}$ which takes domain monics to monics isomorphic to some in $M$, and preserves products, initial object and pullbacks of domain monics, such that $F \simeq P(F')\circ I$.*

Proof. Notice that $F$ preserves domains. Thus $F(\mathrm{dom}\, w\colon X \to X) = (m\colon A \rightarrowtail FX,\, m\colon A \rightarrowtail FX)$. Define $F'(\mathrm{dom}\, w) = K(m\colon A \rightarrowtail FX,\, m\colon A \rightarrowtail FX)$. Next the definition on arrows is forced: $F'(x\colon \mathrm{dom}\, w \to \mathrm{dom}\, z) = h\colon F'(\mathrm{dom}\, w) \to F'(\mathrm{dom}\, z)$, where $h$ is the unique map through which (the representative of) $F(x)$ factors. All the checkings are straightforward.

## 5 - A topos for a dominical category

In this last section we produce an appropriate dominical embedding of a dominical category $\boldsymbol{C}$ into a category of partial maps of a topos. In order to do this, consider the category $\mathrm{Dom}(\boldsymbol{C})$ of domains of $\boldsymbol{C}$. A topos of sheaves on it will do. First take the topos of presheaves: it is well known that the Yoneda embedding preserves all the limit structure, but if we want to preserve the initial object, we must consider just sheaves for the « almost » trivial topology $j$ on $\mathrm{Dom}(\boldsymbol{C})$ where the empty family covers 0. The following statement is straightforward.

Proposition 5.1. *Let $\boldsymbol{E}$ denote the topos $\mathrm{Sh}(\mathrm{Dom}(\boldsymbol{C}), j)$ of sheaves on $\mathrm{Dom}(\boldsymbol{C})$ for the topology $j$. The Yoneda embedding induces a functor $J\colon \mathrm{Dom}(\boldsymbol{C}) \to \boldsymbol{E}$ which preserves limits and the initial object.*

In the sequel we shall make no distinction between $\mathrm{Dom}(\boldsymbol{C})$ and its image through $J$ in $\boldsymbol{E}$.

Let $D$ denote the family of domain maps of $\mathrm{Dom}(\boldsymbol{C})$. As the family of subobject of represented by monics in $D$ is closed under pullbacks in $\mathrm{Dom}(\boldsymbol{C})$, it defines a subfunctor of the subobject classifier $\Omega$ of $\boldsymbol{E}$ as follows: let $\mathrm{dom}\, w$ be any object of $\mathrm{Dom}(\boldsymbol{C})$; define $\Omega_D(\mathrm{dom}\, w)$ as the set of all subobjects of the form $\mathrm{dom}\, z\colon \mathrm{dom}\, z \rightarrowtail \mathrm{dom}\, w$. Pullback along any map $x\colon \mathrm{dom}\, v \to \mathrm{dom}\, w$ applies $\Omega_D(\mathrm{dom}\, w)$ into $\Omega_D(\mathrm{dom}\, v)$. Therefore it gives rise to a subfunctor of $\Omega_D \rightarrowtail \Omega$.

Call a subobject $C \rightarrowtail A$ in $\boldsymbol{E}$ a domain if its classifying map factors through $\Omega_D$.

Remark 5.2. If $A$ is a representable object of $\boldsymbol{E}$, then a domain $C \rightarrowtail A$ is represented by a monic in $D$.

Proposition 5.3. *The subobject $\Omega_D$ contains true and false and is closed under conjunction.*

Proof. Trivial.

Obviously enough, as we have internalized the notion of domain of $\boldsymbol{C}$, we are now able to describe properties of $\boldsymbol{C}$ in the topos $\boldsymbol{E}$. First of all we use this to characterize maps of $\boldsymbol{C}$.

Def. 5.4. Let $A$ be an object of $\boldsymbol{E}$, let $\tilde{A}$ be the object representing partial maps into $A$. Let $\mathrm{Ex}\colon \tilde{A} \to \Omega$ be the classifying map of $A \rightarrowtail \tilde{A}$. Define $\hat{A}$ to be (a representative of) the pullback of $\Omega_D \rightarrowtail \Omega$ along $\mathrm{Ex}\colon \tilde{A} \to \Omega$.

Notice there is a natural injection of $A$ into $\hat{A}$ induced by the pair $A \rightarrowtail \tilde{A}$ and $\mathrm{true} \circ !\colon A \to \Omega$. Instances of this construction can be found in [**4**].

Theorem 5.5. *Let $A$ be an object of $E$. Then $A \rightarrowtail \hat{A}$ classifies partial maps into $A$ defined on a domain.*

Proof. Straightforward.

By Theorem 5.5 any map $w\colon X \to Y$ in $C$ corresponds to a unique map $w''\colon X \to \hat{Y}$, which classifies the partial map $w\colon \mathrm{dom}\, w \to Y$ as $\mathrm{dom}\, w\colon \mathrm{dom}\, w \rightarrowtail X$ is a domain monic.

The last « dominical » construction we consider, which motivated the paper, is that of a Turing morphism. It was introduced in [**2**] and used also in [**3**] under a different name.

Def. 5.6. A map $u\colon X \times X \to X$ in the dominical category $\boldsymbol{C}$ is a *Turing morphism* if it is not a zero map and, for any map $w\colon Y \times X \to X$, there is a total map $f\colon Y \to X$ such that $w = u(f \times \mathrm{id})$ — no request of uniqueness for $f$.

The leading example is the coding of partial recursive functions in the dominical category of partial recursive functions on $N$. We have to point out to the reader that the definition in 5.6 is different from the one given in [**2**], but coincides with it in the case of an « isotypical » category.

In the topos $\boldsymbol{E}$ a Turing morphism does exactly what it is expected to: it enumerates all the partial maps from $X$ into itself.

Theorem 5.7. *A map $u: X \times X \to X$ is a Turing morphism if and only if the map $u': X \to [X \to \hat{X}]$, obtained by exponentially adjoining $u'': X \times X \to \hat{X}$, is an epimorphism in the topos* $\boldsymbol{E}$.

Proof. Suppose $u$ is a Turing morphism: we want to show that $u'$ is pointwise epi. So let $\mathrm{dom}\, z$ be an object of $\mathrm{Dom}(\boldsymbol{C})$, say it is an endomorphism of $Y$ in $C$, and let $w'': \mathrm{dom}\, z \times X \to \hat{X}$ be an element of $[X \to \hat{X}]$. Then $w''$ classifies a partial map $w: \mathrm{dom}\, w \to X$, with $\mathrm{dom}\, w \rightarrowtail \mathrm{dom}\, z \times X \rightarrowtail Y \times X$ a domain monic. Thus $w: Y \times X \to X$ in $\boldsymbol{C}$; hence there is a total map $f: Y \to X$ such that $w = u(f \times \mathrm{id})$. Therefore, set $g$ the map $f \circ \mathrm{dom}\, z: \mathrm{dom}\, z \to X$, it holds that $w = u(g \times \mathrm{id})$ in $\mathrm{Dom}(\boldsymbol{C})$. The map classifying $w$ is then $u''(g \times \mathrm{id})$, yielding that $w'' = u''(g \times \mathrm{id}) = u'(g)$. Conversely, suppose $u'$ is epi. As $\boldsymbol{E}$ is essentially a topos of presheaves over $\mathrm{Dom}(\boldsymbol{C})$ without the initial object, $u'$ must be pointwise epi. Take a map $w: Y \times X \to X$ in $\boldsymbol{C}$. It induces a map $w'': Y \times X \to \hat{X}$ in $\boldsymbol{E}$, that is an element of $[X \to \hat{X}](Y)$. As $u'$ is pointwise epi, $w'' = u'(f)$ for some element $f$ of $X(Y)$. By this we mean a map $f: Y \to X$ in $\mathrm{Dom}(\boldsymbol{C})$, which is a total map $f: Y \to X$ in $\boldsymbol{C}$. As $u''(f \times \mathrm{id}) = u'(f) = w''$, they classify the same map. Hence $w = u(f \times \mathrm{id})$.

As a last remark we notice that a Turing morphism satisfies a stronger property in $\boldsymbol{E}$, as $u'$ is pointwise epi.

Corollary 5.8. *Let $u: X \times X \to X$ be a Turing morphism. Then «composition with $u'$ on the left» is epi for all representables, that is the map $u' \circ -: [A \to X] \to [A \to [X \to \hat{X}]]$ for all representable functors $A$ in* $\boldsymbol{E}$.

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## Sommario

*Data una categoria dominicale* **C**, *si introduce una opportuna categoria di « domini » per rappresentare* **C** *come categoria di mappe parziali. Si riesce a dare una proprietà universale che caratterizza la costruzione. Quindi si studiano i morfismi di Turing in un topos che estende la categoria di domini, mostrando come essi possano essere considerati enumerazioni delle funzioni parziali dell'oggetto in sè.*


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