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On the solutions of the linear Maxwell-Boltzmann equation (**)

1 - Introduction

We consider a mixture of two different types of particles (called M and N) in the physical space. We suppose that particles M verify the following assumptions: they condition the physical behaviour of type N, without being conditioned themselves; their distribution function is known, but not necessarily maxwellian.

As for particles N, we suppose that every type N particle (with mass m) is acted upon by an external force F (that depends on the position vector r, on the velocity v of the same particle and also on time t); that the collisions of particles N amongst themselves can be overlooked compared to those with particles of type M.

Now, $f(r, v, t)$, probability density at time t in the phase space of a single N particle, satisfies, as is known [**5**], the linear Maxwell-Boltzmann equation

$$(1)\qquad \left(\frac{\partial f}{\partial t} + v\cdot\frac{\partial f}{\partial r} + \frac{F}{m}\cdot\frac{\partial f}{\partial v} + \frac{1}{m}(\mathrm{div}_v F)f\right)(r, v, t)$$
$$= -(\nu f)(r, v, t) + \int_{R^3} k(r, v, v', t) f(r, v', t)\,\mathrm{d}v' \qquad t\in R\,, \quad (r, v)\in R^6,$$

where

$$v\cdot\frac{\partial f}{\partial r} = \sum_{i=1}^{3} v_i\frac{\partial f}{\partial r_i}, \qquad F\cdot\frac{\partial f}{\partial v} = \sum_{i=1}^{3} F_i\frac{\partial f}{\partial v_i}, \qquad \mathrm{div}_v F = \sum_{i=1}^{3}\frac{\partial F_i}{\partial v_i},$$

ν and k represent respectively collision frequency and scattering Kernel

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The great importance of this equation in the sphere of the kinetic theory of gases and in particular of plasmas is known.

We mention, for example, the studies carried out, starting from the above mentioned equation, even if in its integral formulation, by Ganapol and Boffi [**7**]$_2$ on electrical conductivity and by Boffi, Molinari and Scardovelli [**3**] on the propagation of electromagnetic waves in a weakly ionized plasma. All this shows the great interest of the study of equation (1) and in particular of the corresponding Cauchy problem. Nevertheless until now this problem has received very little attention: in fact the greater part of research carried out on the Boltzmann equation concerns the non-maxwellian case, that is the case $F = 0$. As for as we know, the initial value problem for (1) has been considered in [**7**]$_1$ by Ganapol and Boffi, supposing F to be constant, in [**4**] by Boffi and Nonnenmacher, in [**9**] by Molinet, supposing f to be space-independent and in [**6**] by Drange, supposing F to be the Lorentz face. Recently with techniques which are different from those used in the mentioned works, that is by using the linear semigroup theory, Arlotti, in a previous paper [**1**]$_2$, studied equation (1) by supposing that the external force, the collision frequency and the scattering Kernel depend on the position vector and on the velocity, but not on time.

The aim of this research is a further study of the Cauchy problem for eq. (1), which will be carried out, now supposing that functions F, ν and k are dependent also on time t. More precisely here we will suppose that at least the following assumptions are satisfied:

(i)$_1$ $F: R^7 \to R^3$ is continuous;

(i)$_2$ there exists a continuous function $\chi: R \to R^+$ (set of all real non negative numbers) such that $\forall t \in R$, $r, v, r', v' \in R^3$ it results

$$|F(r, v, t) - F(r', v', t)| \leqq \chi(t)\,|(r, v) - (r', v')|$$

(here and following each R^n, $n \geqq 1$, is thought to be provided with the usual norm which, without risk of misunderstanding, will, in every case, be indicated by $|\cdot|$);

(i)$_3$ there exists a continuous function $\nu_0: R \to R^+$ such that $\forall t \in R$, $r, v \in R^3$ it results $0 \leqq \nu(r, v, t) \leqq \nu_0(t)$;

(i)$_4$ the map $R^7 \ni (r, v, t) \to ((1/m)\,\mathrm{div}_v F + \nu)(r, v, t) \in R^+$ is continuous;

(i)$_5$ $\forall t \in R$ the mapping $R^9 \ni (r, v, v') \to (r, v, v', t) \in R^+$ is measurable;

(i)$_6$ $\forall (r, v', t) \in R^7$ it is $\int_{R^3} k(r, v, v', t)\,dv = \nu(r, v', t)$;

(i)$_7$ $\forall \tau \in R$ it is $\lim_{t \to \tau} \int_{R^3} |k(r, v, v', t) - k(r, v, v', \tau)|\,dv = 0$ for a.e. $(r, v') \in R^6$.

These assumptions appear to be natural, considering the physical meaning of functions F, ν and k. As for unknown f, it must be, as a distribution function, non negative, Lebesgue integrable, but not necessarily continuously differentiable on the phase space R^6. Therefore we are brought to consider f, in the equation in question, as an unknown function of time with values in Banach space $X = L^1(R^6)$. As a consequence of this, in the following section **2** we will formulate the Cauchy problem for the equation (1) in abstract form in space X.

Section **3** and **4** are dedicated to a study of the linear operators $A(t)$ introduced in **2** (see def. (4)).

Through this study in **5**, using the evolution operator theory, we will establish an existence and uniqueness theorem of the weak solution for the Cauchy problem relative to eq. (1). Finally in **6** we shall see that, if further adequate regularity assumptions for F, ν and k are verified, then the above mentioned solution is also a strong solution.

2 - Abstract formulation

To formulate in abstract form the Cauchy problem for eq. (1), we consider the Banach space $X = L^1(R^6)$, provided with the usual norm, which we will indicate by $\|\cdot\|$. We then observe that the left side of (1) can be written in the more compact form

$$\left(\frac{\partial f}{\partial t} + a\cdot\frac{\partial f}{\partial x}\right)(x, t) \quad \text{where} \quad a\cdot\frac{\partial f}{\partial x} = \sum_{h=1}^{6} a_h \frac{\partial f}{\partial x_h} \qquad x = (r, v) \in R^6\,, \quad t \in R$$

(the meaning of a, a_h, x_h being obvious).

Owing to $(\mathrm{i})_1$ a is a known, continuous function defined on R^7 with values on R^6. Because of $(\mathrm{i})_2$ there is also a continuous function $\mu\colon R \to R^+$ such that

$$|a(x, t) - a(y, t)| \leqq \mu(t)\,|x - y| \qquad \forall x, y \in R^6\,, \quad t \in R\,. \tag{2}$$

This allows us to define the following family of linear operators $A(t)$.

We put $\forall t \in R$, $f \in X$

$$\mathscr{A}(t) f = a\cdot\frac{\partial f}{\partial x}\,, \tag{3}$$

where, because of (2) and of a Rademacher theorem [**10**], the right side of (3) makes sense as a distribution. For every $t \in R$ we indicate with $A(t)$ the linear operator defined by

$$(4)\qquad D(A(t)) = \{f \in X \colon \mathscr{A}(t) f \in X\}\,, \qquad A(t) f = \mathscr{A}(t) f\,, \qquad R(A(t)) \subseteq X\,.$$

We then define the families of linear operators $B(t), K(t)$ and $L(t)$ $(t \in R)$, by putting

$$(5)\qquad \begin{aligned} &D(B(t)) = X && R(B(t)) \subseteq X\,, \\ &(B(t)f)(x) = -(\nu + \frac{1}{m}\operatorname{div}_v F)(x,t) f(x) = b(x,t) f(x) && x = (r, v) \in R^6\,. \end{aligned}$$

$$(6)\qquad \begin{aligned} &D(K(t)) = X && R(K(t)) \subseteq X, \\ &(K(t)f)(x) = \int_{R^3} k(r, v, v', t) f(r, v')\, dv' && x = (r, v) \in R^6. \end{aligned}$$

$$(7)\qquad \begin{aligned} &D(L(t)) = X && R(L(t)) \subseteq X\,, \\ &L(t) f = B(t) f + K(t) f\,. \end{aligned}$$

We can immediately see, because of assumptions $(i)_2$-$(i)_7$, that definitions (5), (6) and (7) are correct and that, for every $t \in R$, $B(t)$, $K(t)$ and $L(t)$ are bounded linear operators. If $b_0 \colon R \to R^+$ is a continuous function such that it results $|b(x,t)| \leqq b_0(t)$ $\forall (x,t) \in R^7$, then we have $\|B(t)\| \leqq b_0(t)$. We have also $\|K(t)\| \leqq \nu_0(t)$ and therefore $\|L(t)\| \leqq b_0(t) + \nu_0(t)$. We can finally see that the mappings $t \to B(t)$, $t \to K(t)$ and also $t \to L(t)$ from R to $\mathscr{B}(X)$ (set of all bounded linear operators from X to X) are strongly continuous.

By using the previous notations, the Cauchy problem for the equation (1) can be written in abstract form in the following way

$$(18)\qquad \frac{df}{dt} = -A(t) f + L(t) f \qquad t > 0\,, \qquad f(0) = f_0 \geqq 0\,,$$

where the unknown f is now a mapping of R^+ into X, d/dt is a strong derivative.

3 - The study of the linear operators $A(t)$. Preliminaries

As explained in the previous section, let $\boldsymbol{a} \colon R^7 \to R^6$ be a continuous function which verifies condition (2).

It is known (see [**13**]) that such a function possesses almost everywhere on R^7 the first order partial derivatives $\partial \boldsymbol{a}/\partial x_h$ $(h = 1, \dots, 6)$, which are measurable and bounded, giving for every h and for a.e. $(x,t) \in R^7$, $|(\partial \boldsymbol{a}/\partial x_h)(x,t)|$

$\leq \mu(t)$. Therefore, if we put $\operatorname{div} a = \sum_{h=1}^{6} \partial a_h / \partial x_h$, then there certainly exists a continuous function $\bar{\mu}: R \to R^+$ such that

$$|\operatorname{div} a(x, t)| \leqq \bar{\mu}(t) \qquad \text{for a.e. } (x, t) \in R^7 . \tag{9}$$

It is also known that, for such a function a and $\forall (x, t) \in R^7$, the initial value problem

$$\frac{dy}{ds} = a(y, s) \qquad y \in R^6,\ s \in R; \qquad y(t) = x \quad x \in R^6,\ t \in R \tag{10}$$

has one and only one (classical) solution.

This allows us to consider the function $\varphi: R^8 \to R^6$ defined by the condition that $\forall (x, t) \in R^7$ $s \to \Phi(x, t, s)$, $s \in R$ is the only solution of the problem (10).

We now list the properties of function Φ which will later be useful; they are proved in [**8**], [**13**], [**14**] and [**1**]$_1$.

Properties of Φ:

$(\alpha)_1$ $\Phi(x, t, t) = x \qquad \forall x \in R^6, \quad t \in R.$

$(\alpha)_2$ $\Phi(\Phi(x, t, s), s, \sigma) = \Phi(x, t, \sigma) \qquad \forall x \in R^6, \quad \sigma, s, t \in R.$

$(\alpha)_3$ $\forall s, t \in R$ the transformation $x \to \Phi(x, t, s)$ of R^6 into R^6 is measurable, invertible with the measurable inverse $x \to \Phi(x, s, t)$.

$(\alpha)_4$ $|\Phi(x, t, s) - x| \leqq h(t, s)|x| + k(t, s) \quad \forall x \in R^6, \quad s, t \in R$

if $$h(t, s) = \int_{s\wedge t}^{s\vee t} \mu(\tau)\Big[\exp \int_{\tau\wedge s}^{\tau\vee s} \mu(\sigma)\, d\sigma\Big]\, d\tau , \qquad k(t, s) = \int_{s\wedge t}^{s\vee t} |a(0, \tau)|\Big[\exp \int_{\tau\wedge s}^{\tau\vee s} \mu(\sigma)\, d\sigma\Big]\, d\tau .$$

$(\alpha)_5$ $$|\Phi(x, t, s)| \leqq \Big(\exp \int_{s\wedge t}^{s\vee t} \omega(\sigma)\, d\sigma\Big)\big(|x| + 1\big) \quad \forall x \in R^6,\ s, t \in R^6$$

if $$\omega(t) = \mu(t) \vee |a(0, t)| \qquad \forall t \in R .$$

$(\alpha)_6$ $$|x| \leqq \Big(\exp \int_{s\wedge t}^{s\vee t} \omega(\sigma)\, d\sigma\Big)\big(|\Phi(x, t, s)| + 1\big) \quad \forall x \in R^6, \quad s, t \in R.$$

$(\alpha)_7$ $$|\Phi(x, t, s) - \Phi(x, \tau, s)| \leqq [h(t, \tau)|x| + k(t, \tau)]\Big(\exp \int_{s\wedge\tau}^{s\vee\tau} \mu(\sigma)\, d\sigma\Big)$$

$$\forall x \in R^6 , \qquad s, t, \tau \in R .$$

$(\alpha)_8$ $\quad |\Phi(x,t,s)-\Phi(x,t,\sigma)| \leqq \big[\int\limits_{s\wedge\sigma}^{s\vee\sigma}\omega(\tau)\big(\exp\int\limits_{t\wedge\tau}^{t\vee\tau}\omega(t')\,dt'\big)\,d\tau\big]\big(|x|+1\big)$

$$\forall x\in R^6\,,\qquad s,\sigma,t\in R\,.$$

$(\alpha)_9$ $\quad |x-y|\,\exp\int\limits_{s\vee t}^{s\wedge t}\mu(\sigma)\,d\sigma \leqq |\Phi(x,t,s)-\varphi(y,t,s)| \leqq |x-y|\,\exp\int\limits_{s\wedge t}^{s\vee t}\mu(\sigma)\,d\sigma$

$$\forall x,y\in R^6\,,\qquad s,t\in R\,.$$

$(\alpha)_{10}$ The map $R^8\ni(x,t,s)\to\Phi(x,t,s)\in R^6$ is continuous.

$(\alpha)_{11}$ $\forall t\in R$ the transformation $(x,s)\to\big(\Phi(x,t,s),s\big)$ of R^7 into R^7 is measurable, invertible with the measurable inverse $(x,s)\to\big(\Phi(x,s,t),s\big)$.

$(\alpha)_{12}$ $\forall(x,t,s)\in R^8$ we have $(\partial\Phi/\partial s)(x,t,s)=a\big(\Phi(x,t,s),s\big)$.

$(\alpha)_{13}$ There exists a measurable subset E of R^7 which has the following properties: $m(R^7\setminus E)=0$; $\forall t\in R$ cross-sections $\{E_t=x\colon(x,t)\in E\}$ are measurable, having $m(R^6\setminus E_t)=0$; finally at each point $(x,t,s)\in R^8$ so that $(x,t)\in E$ the following first order partial derivatives exist $(\partial\Phi/\partial x_h)(x,t,s)$ $(h=1,\dots,6)$. For any $h,k=1,\dots,6$, are all measurable the real-valued functions

$$(x,t,s)\to\frac{\partial\Phi_k}{\partial x_h}(x,t,s)\,,\qquad (x,t)\to\frac{\partial\Phi_k}{\partial x_h}(x,t,s)\qquad s\in R \text{ arbitrarily fixed;}$$

$$(x,s)\to(\partial\Phi_k/\partial x_h)(x,t,s)\qquad t\in R \text{ arbitrarily fixed;}$$

$$x\to(\partial\Phi_k/\partial x_h)(x,t,s),\quad s,t\in R \text{ arbitrarily fixed.}$$

$(\alpha)_{14}$ If we put $J(x,t,s)=\det\big((\partial\Phi_k/\partial x_h)(x,t,s)\big)$, then we have

$$\forall(x,t,s)\in E\times R\qquad J(x,t,s)=\exp\int\limits_t^s\operatorname{div}a\big(\Phi(x,t,\sigma),\sigma\big)\,d\sigma\,.$$

$(\alpha)_{15}$ $\forall(x,t,s)\in E\times R$ it is $\exp\int\limits_{s\vee t}^{s\wedge t}\bar\mu(\sigma)\,d\sigma\leqq J(x,t,s)\leqq\exp\int\limits_{s\wedge t}^{s\vee t}\bar\mu(\sigma)\,d\sigma$.

4 - Evolution operators generated by family $\{A(t)\}$

We now put $\forall s,t\in R$, $f\in X$

$$U_0(t,s)f=f\big(\Phi(\cdot,t,s)\big)\,. \tag{11}$$

After having realized that (11) defines a family of linear operators from X to X, we want to list the more important properties of this family; they can be easily shown with techniques similar to those used in [**1**]$_1$.

$(\beta)_1$ $\forall s, t \in R \quad U_0(t, s)$ is a bounded linear operator from X to X.

$(\beta)_2$ $\exp\int_{s\vee t}^{s\wedge t} \bar{\mu}(\sigma)\,d\sigma \leqq \|U_0(t, s)\| \leqq \exp\int_{s\wedge t}^{s\vee t} \bar{\mu}(\sigma)\,d\sigma \quad \forall s, t \in R.$

$(\beta)_3$ $U_0(t, t)f = f \quad \forall t \in R,\ f \in X.$

$(\beta)_4$ $U_0(t, s)\,U_0(s, \sigma)f = U_0(t, \sigma)f \quad \forall \sigma, s, t \in R,\ f \in X.$

$(\beta)_5$ $\lim_{(t,s)\to(\tau,\sigma)} U_0(t, s)f = U_0(\tau, \sigma)f \quad \forall \sigma, \tau \in R,\ f \in X.$

$(\beta)_6$ If $f \in X$ and $|x|f \in X$, then $|x|\,U_0(t, s)f \in X\ \forall s, t \in R.$
Moreover we have

$$\|\,|x|\,U_0(t, s)f\| \leqq \Big[\exp\int_{s\wedge t}^{s\vee t} \big(\omega(\sigma) + \bar{\mu}(\sigma)\big)\,d\sigma\Big]\big[\|\,|x|f\| + \|f\|\big] .$$

$(\beta)_7$ If $f \in X$ with $|x|f \in X$, then $\forall \sigma, \tau \in R$ we have

$$\lim_{(t,s)\to(\tau,\sigma)} \int_{R^6} |x|\,\big|f\big(\Phi(x, t, s)\big) - f\big(\Phi(x, \tau, \sigma)\big)\big|\,dx = 0 .$$

$(\beta)_8$ If $f \in X$ with $|x|f \in X$, then $\forall \sigma, \tau \in R$ we have

$$\lim_{t\to\tau} \int_{R^6} \big|\big(a_h(x, t) - a_h(x, \tau)\big) f\big(\Phi(x, t, \sigma)\big)\big|\,dx = 0 \qquad h = 1, \dots, 6 .$$

$(\beta)_9$ If $f \in W^{1,1}(R^6) = \{f \in X;\ \partial f/\partial x_h \in X,\ h = 1, \dots, 6\}$ (we intend the derivatives to be in distribution sense) then $\forall s, t \in R$ we have $U_0(t, s)f \in W^{1,1}(R^6)$.

Proof. We first suppose $f \in C_0^1(R^6)$. Then, for $(\beta)_1$, $f\big(\Phi(\cdot, t, s)\big) \in X$; for $(\alpha)_9$ $x \to f\big(\Phi(x, t, s)\big)$ is Lipschitz continuous; for $(\alpha)_{13}$ $\forall x \in E_t$ there exist first order partial derivatives $(\partial/\partial x_h) f\big(\Phi(x, t, s)\big)$ $h = 1, \dots, 6$ given by the formula

$$\frac{\partial}{\partial x_h} f\big(\Phi(x, t, s)\big) = \sum_{k=1}^{6} \frac{\partial f}{\partial y_k}\big(\Phi(x, t, s)\big)\frac{\partial \Phi_k}{\partial x_h}(x, t, s) \qquad y = \Phi(x, t, s) . \tag{12}$$

It is obvious that, as a consequence of the properties of f and Φ, the right side of (12) (and therefore also the left side) is Lebesgue integrabe on R^6; so $(\partial/\partial x_h) f\big(\Phi(x, t, s)\big) \in X\ \forall h$.

All this proves (see [**11**], p. 57) that $U_0(t, s)f \in W^{1,1}(R^6)$ and that formula (12) is true also in distribution sense. In this way it results for such an f and $\forall \psi \in C_0^\infty(R^6)$

$$-\int_{R^6} f\big(\Phi(x, t, s)\big)\frac{\partial \psi}{\partial x_h}(x)\,dx = \int_{R^6} \sum_{k=1}^{6} \frac{\partial f}{\partial y_k}\big(\Phi(x, t, s)\big)\frac{\partial \Phi_k}{\partial x_h}(x, t, s)\,\psi(x)\,dx . \tag{13}$$

Replacing x by $\Phi(y, s, t)$ and considering $(\alpha)_3$, $(\alpha)_{13}$, $(\alpha)_{14}$ and $(\alpha)_{15}$ we can easily see that (13) is equivalent to

$$(14)\qquad -\int_{R^6} f(y)\,\frac{\partial\psi}{\partial x_h}\,(\Phi(y, s, t))J(y, s, t)\,\mathrm{d}y$$

$$= \int_{R^6} \sum_{k=1}^{6} \frac{\partial f}{\partial y_k}(y)\,\frac{\partial \Phi_k}{\partial x_h}\,(\Phi(y, s, t), t, s)\psi(\Phi(y, s, t))J(y, s, t)\,\mathrm{d}y\,.$$

From this result, we can immediately recognize the truth of the thesis in the general case (see [**2**], p. 64).

$(\beta)_{10}$ If $f \in \mathscr{D}(R^6) = \{f \in W^{1,1}(R^6):\ |x|\,\partial f/\partial x_h \in X,\ h = 1, \ldots, 6\}$, then $\forall s, t \in R$ we have $U_0(t, s) f \in \mathscr{D}(R^6)$.

$(\beta)_{11}$ If $f \in \mathscr{E}(R^6) = \{f \in \mathscr{D}(R^6):\ |x| f \in X\}$, then $\forall s, t \in R$ we have $U_0(t, s) f \in \mathscr{E}(R^6)$.

$(\beta)_{12}$ $\forall f \in \mathscr{D}(R^6)$, $\tau \in R$ it results $\lim_{t\to\tau} A(t) f = A(\tau) f$.

$(\beta)_{13}$ $\forall f \in \mathscr{D}(R^6)$, $\sigma, \tau \in R$ it results $\lim_{(t,s)\to(\tau,\sigma)} U_0(t, s) A(s) f = U_0(\tau, \sigma) A(\sigma) f$.

$(\beta)_{14}$ $\forall f \in \mathscr{D}(R^6)$ $\tau \in R$ the first order partial derivatives (in the distributional sense) of the function $(x, s) \to f(\Phi(x, \tau, s))$ $((x, s) \in R^7)$ are functions and also integrable according to Lebesgue on any subset of R^7 of the form $R^6 \times I$ (I being a bounded interval), they are obtained by applying the usual differentiation laws.

Proof. It is similar to that of $(\beta)_9$.

$(\beta)_{15}$ $\forall f \in \mathscr{D}(R^6)$, $\sigma, \tau \in R$ it results

$$\lim_{h\to 0} \left\| \frac{U_0(\tau, \sigma + h) f - U_0(\tau, \sigma) f}{h} - U_0(\tau, \sigma) A(\sigma) f \right\| = 0\,.$$

$(\beta)_{16}$ $\forall f \in \mathscr{D}(R^6)$, $\sigma, \tau \in R$ it results

$$\lim_{h\to 0} \left\| \frac{U_0(\tau + h, \sigma) f - U_0(\tau, \sigma) f}{h} + A(\tau) U_0(\tau, \sigma) f \right\| = 0\,.$$

As a consequence of the above listed properties, the following is true.

Theorem 1. *If assumptions* (i)$_1$ *and* (i)$_2$ *are true, then for each pair of real numbers* s, t *there exists an operator* $U_0(t, s)$ *belonging to the set* $\mathscr{B}(X)$ *and pos-*

sessing the following properties:

$(\gamma)_1$ $\exp \int_{s\vee t}^{s\wedge t} \bar{\mu}(\sigma)\,d\sigma \leqq \| U_0(t, s) \| \leqq \exp \int_{s\wedge t}^{s\vee t} \bar{\mu}(\sigma)\,d\sigma \ \ \forall s, t \in R.$

$(\gamma)_2$ $U_0(t, t) = I \ \ \forall t \in R.$

(γ_3) $U_0(t, \sigma)\, U_0(\sigma, s) = U_0(t, s) \ \ \forall \sigma, s, t \in R.$

$(\gamma)_4$ *The map* $(t, s) \to U_0(t, s)$ *from* R^2 *to* $\mathscr{B}(X)$ *is strongly continuous.*

$(\gamma)_5$ $(\partial/\partial s)\, U_0(t, s) f = U_0(t, s) A(s) f \ \ \forall f \in \mathscr{D}(R^6), \ \ s, t \in R$ *and the map*

$R^2 \ni (t, s) \to U_0(t, s) A(s) f \in X$ *is strongly continuous.*

$(\gamma)_6$ $(\partial/\partial t)\, U_0(t, s) f = -A(t)\, U_0(t, s) f \ \ \forall f \in \mathscr{D}(R^6), \ \ s, t \in R.$

We call the family of operators $\{U_0(t, s)\}$ the evolution operators generated by $\{A(t)\}$.

5 - Weak solution

In virtue of the theorem established in the previous section, we can state that every eventual strongly continuous solution of problem (8) is also a solution of the abstract integral equation

$$f(t) = U_0(t, 0) f_0 + \int_0^t U_0(t, s) L(s) f(s)\, ds. \tag{15}$$

Considering (15), the following holds.

Theorem 2. *For every* $f_0 \in X$ *there exists one and only one function* $t \to f(t)$ *defined on* R^+ *with values in* X *which is strongly continuous and which verifies integral equation* (15). *If* $f_0 \in X^+$, *that is if* $f_0 \in X$ *and* $f_0(x) \geqq 0$ *for a. e.* $x \in R^6$, *then also* $f(t) \in X^+$ $\forall t > 0$.

For the proof of Theorem 2 we will use the following perturbation lemma which is independent from the preceding assumptions and physical motivations.

Lemma 1. *Let* $\{Z_0(t, s)\}$ *be a family of bounded linear operators from* X *to* X *with the following properties*

$$Z_0(t, t) = I \quad \forall t \in R; \qquad Z_0(t, \sigma) Z_0(\sigma, s) = Z_0(t, s) \quad \forall \sigma, s, t \in R;$$

the map $R^2 \ni (t, s) \to Z_0(t, s) \in \mathscr{B}(X)$ *is strongly continuous;* $\|Z_0(t, s)\| \leqq g_0(s\wedge t, s\vee t)$, g_0 *being a continuous function with real positive values, which satisfies the functional equation*

$$g_0(s, \sigma)g_0(\sigma, t) = g_0(s, t) \qquad \forall s \leqq \sigma \leqq t .$$

Moreover let $\{C(t)\}$ *be a family of bounded linear operators from* X *to* X, *so that* $C(t)$ *is strongly continuous in* t *and* $\|C(t)\| \leqq \psi(t)$ ($t \to \psi(t)$ *being a continuous function with real positive values*).

Then there exists one and only one family $\{Z(t, s)\}$ *so that function* $R^2 \ni (t, s) \to Z(t, s) \in \mathscr{B}(X)$ *is a strongly continuous solution of the abstract integral equation*

$$Z(t, s) = Z_0(t, s) + \int_s^t Z_0(t, \sigma)\, C(\sigma)\, Z(\sigma, s)\, d\sigma .$$

Proof. By putting

$$Z_n(t, s) = \int_s^t Z_0(t, \sigma)\, C(\sigma) Z_{n-1}(\sigma, s)\, d\sigma \qquad n = 1, 2, \dots ,$$

it is easy to recognize by induction that $\forall n \in N$, $s, t \in R$ we have $Z_n(t, s) \in \mathscr{B}(X)$ with

$$\|Z_n(t, s)\| \leqq g_0(s\wedge t, s\vee t)\, \frac{\left(\int_{s\wedge t}^{s\vee t} \psi(\sigma)\, d\sigma\right)^n}{n!} ;$$

moreover $\forall n \in N$ the map $(t, s) \to Z_n(t, s)$ is strongly continuous. If then makes sense to put

$$Z(t, s) = \sum_{n=0}^{\infty} Z_n(t, s) ,$$

since the right side series is strongly convergent, uniformly in (t, s) on every bounded rectangle.

Obviously $Z(t, s)$ has the above mentioned properties; it is moreover $Z(t, t) = I$ $\forall t \in R$; $Z(t, \sigma)\, Z(\sigma, s) = Z(t, s)$ $\forall \sigma, s, t \in R$ (see [**13**] p. 100); $\|Z(t, s)\| \leqq g(s\wedge t, s\vee t)$, if we put for $s \leqq t$

$$g(s, t) = g_0 \cdot (s, t) \exp \int_s^t \psi(\sigma)\, d\sigma .$$

We are now able to prove Theorem 2.

Proof. of Theorem 2. Let $\{U(t,s)\}$ be the family of bounded linear operators from X to X defined by

$$(16)\qquad U(t,s)=\sum_{n=0}^{\infty}U_n(t,s)\,,\qquad U_n(t,s)=\int_s^t U_0(t,\sigma)L(\sigma)U_{n-1}(\sigma,s)\,d\sigma\qquad n=1,2\ldots.$$

Furthermore $\forall t\geqq 0$ we put $f(t)=U(t,0)f_0$.

Because of Lemma 1 we can state that $t\to f(t)$ is a strongly continuous map of R^+ into X and that this function is the only continuous solution of integral equation (15). We now want to show that if $f_0\in X^+$, then also $f(t)\in X^+$. To do this we consider the two families $\{U_\nu(t,s)\}$ and $\{T(t,s)\}$ given by

$$U_\nu(t,s)=\sum_{n=0}^{\infty}S_n(t,s)\,,\qquad T(t,s)=\sum_{n=0}^{\infty}T_n(t,s)\,,$$

with

$$S_0(t,s)=U_0(t,s)\,,\qquad S_n(t,s)=\int_s^t U_0(t,\sigma)B(\sigma)S_{n-1}(\sigma,s)\,d\sigma\qquad n=1,2,\ldots,$$

$$T_0(t,s)=U_\nu(t,s)\,,\qquad T_n(t,s)=\int_s^t U_\nu(t,\sigma)K(\sigma)T_{n-1}(\sigma,s)\,d\sigma\qquad n=1,2,\ldots.$$

In virtue of (5) and $(\mathrm{i})_4$, it is easy to see, by induction, that $\forall f\in X$ it results

$$\big(S_n(t,s)f\big)(x)=\frac{1}{n!}\Big[\int_s^t b\big(\Phi(x,t,\sigma),\sigma\big)\,d\sigma\Big]^n f\big(\Phi(x,t,s)\big)\qquad n=1,2,\ldots$$

and therefore

$$(17)\qquad \big(U_\nu(t,s)f\big)(x)=\exp\Big(\int_s^t b\big(\Phi(x,t,\sigma),\sigma\big)\,d\sigma\Big)\,f\big(\Phi(x,t,s)\big)\,.$$

This result proves that if $f\in X^+$ then also $U_\nu(t,s)f\in X^+$, $\forall s,t\in R$. Because of this, definition (6) and assumption $(\mathrm{i})_5$ we can state that if $f\in X^+$ and $s\leqslant t$, then also $T(t,s)f\in X^+$.

To complete the proof of Theorem 2 it is enough to show that $T(t,s)=U(t,s)$ $\forall s,t\in R$.

In fact, in virtue of (7), we obtain by recursion $\forall n\in N$, $s,t\in R$

$$U_n(t,s)=S_n(t,s)+\int_s^t\sum_{k=0}^{n-1}S_k(t,\sigma)K(\sigma)U_{n-1-k}(\sigma,s)\,d\sigma\,.$$

But $\sum_{k=0}^{n-1}S_k(t,\sigma)K(\sigma)U_{n-1-k}(\sigma,s)$ is the general term of the Cauchy product of two series $\sum_{n=0}^{\infty}S_n(t,\sigma)$ and $\sum_{n=0}^{\infty}K(\sigma)U_n(\sigma,s)$, which are totally convergent as

σ, s and t vary in a bounded interval. Therefore, for these σ, s and t, also the product is totally convergent; furthermore its sum is $U_\nu(t,\sigma)K(\sigma)U(\sigma,s)$. Because of Lemma 2 the thesis is now immediate.

We should like to point out that family $\{U(t,s)\}$ possesses properties $(\delta)_1$-$(\delta)_5$ listed in the following Theorem 3.

6 - Strong solution

The aim of this section is to show that, if the external force, the collision frequency and the scattering Kernel verify not only conditions $(i)_1$–$(i)_7$, but also further adequate regularity assumptions, then, for convenient initial f_0, Cauchy problem (8) has a strong solution. More precisely, we will consider F in such a way as to verify the further condition

(j)$_1$ $\forall h = 1, \ldots, 6$ and $\forall(x,t)\in R^7$ there exist the partial derivatives $(\partial\alpha/\partial x_h)(x,t)$ and the maps $(x,t)\to(\partial\alpha/\partial x_h)(x,t)$ are continuous from R^7 into R^6.

Regarding b and k, we will suppose that they verify either the further conditions (j)$_2$→ (j)$_3$ and (j)$_4$ or conditions (j)$'_2$, (j)$'_3$ and (j)$'_4$.

(j)$_2$ $\forall t\in R$, $f\in\mathscr{D}(R^6)$ we have $L(t)f\in\mathscr{D}(R^6)$.

(j)$_3$ There exists a continuous function $l_1\colon R\to R^+$ such that $\forall t\in R$, $f\in\mathscr{D}(R^6)$ it results

$$\sum_{h=1}^{6}\Big[\Big\|\frac{\partial}{\partial x_h}L(t)f\Big\| + \Big\||x|\frac{\partial}{\partial x_h}L(t)f\Big\|\Big] \leqq l_1(t)\Big[\|f\| + \sum_{h=1}^{6}\Big(\Big\|\frac{\partial f}{\partial x_h}\Big\| + \Big\||x|\frac{\partial f}{\partial x_h}\Big\|\Big)\Big].$$

(j)$_4$ If $(t,s)\to Z(t,s)$ is a strongly continuous function defined on R^2 with values in $\mathscr{B}(X)$ so that

(i) if $f\in\mathscr{D}(R^6)$ then also $Z(t,s)f\in\mathscr{D}(R^6)$ and the maps $(t,s)\to(\partial/\partial x_h)Z(t,s)f$, $(t,s)\to|x|(\partial/\partial x_h)Z(t,s)f$ $(h=1,\ldots,6)$ of R^2 into X are strongly continuous, then also $(t,s)\to L(t)Z(t,s)$ is such a function.

(j)$'_2$ $\forall t\in R$, $f\in\mathscr{E}(R^6)$ we have $L(t)f\in\mathscr{E}(R^6)$.

(j)$'_3$ There exist two continuous functions $\lambda\colon R\to R^+$ and $l_2\colon R\to R^+$ (with $\lambda \geqq b_0+\nu_0$) such that $\forall t\in R$, $f\in\mathscr{E}(R^6)$ it results

$$\||x|L(t)f\| \leqq \lambda(t)[\|f\| + \||x|f\|]$$

$$\sum_{h=1}^{6}\Big[\Big\|\frac{\partial}{\partial x_h}L(t)f\Big\| + \Big\||x|\frac{\partial}{\partial x_h}L(t)f\Big\|\Big]$$

$$\leqq l_2(t)\Big[\|f\| + \||x|f\| + \sum_{h=1}^{6}\Big(\Big\|\frac{\partial f}{\partial x_h}\Big\| + \Big\||x|\frac{\partial f}{\partial x_h}\Big\|\Big)\Big].$$

$(\mathrm{j})_4'$ if $(t, s) \to Z(t, s)$ is a strongly continuous function defined on R^2 with values in $\mathscr{B}(X)$, so that

(i)$'$ if $f \in \mathscr{E}(R^6)$ then also $Z(t, s)f \in \mathscr{E}(R^6)$ and the maps

$$(t, s) \to |x|\, Z(t, s) f\,, (t, s) \to \frac{\partial}{\partial x_h} Z(t, s) f, \quad (t, s) \to |x|\, \frac{\partial}{\partial x_h}\, Z(t, s) f \qquad h = 1\,, \ldots, s$$

of R^2 into X are strongly continuous, then also $(t, s) \to L(t)G(t, s)$ is such a function.

Having considered this, we can see the truth of the following

Theorem 3. *If all the assumptions* $(\mathrm{i})_1$-$(\mathrm{i})_7$, $(\mathrm{j})_1$ *and either assumptions* $(\mathrm{j})_2$-$(\mathrm{j})_4$ *or* $(\mathrm{j})_2'$-$(\mathrm{j})_4'$ *are true, then the family of bounded linear operators* $\{U(t, s)\}$ *defined by* (16) *has the following properties:*

$(\delta)_1$ $\quad \|U(t, s)\| \leq \exp \int_{s\wedge t}^{s\vee t} 2\nu_0(\sigma)\, d\sigma \quad \forall s, t \in R.$

$(\delta)_2$ $\quad U(t, t) = I \quad \forall t \in R.$

$(\delta)_3$ $\quad U(t, \sigma)\, U(\sigma, s) = U(t, s) \quad \forall \sigma, s, t \in R.$

$(\delta)_4$ $\quad$ *The map* $(t, s) \to U(t, s)$ *from* R^2 *to* $\mathscr{B}(X)$ *is strongly continuous.*

$(\delta)_5$ $\quad (\partial/\partial s)\, U(t, s) f = U(t, s)\big(A(s) - L(s)\big) f \quad \forall s, t \in R \quad f \in \mathscr{D}(R^6)$, *and the map* $R^2 \ni (t, s) \to U(t, s)\big(A(s) - L(s)\big) f \in X$ *is strongly continuous.*

$(\delta)_6$ $\quad (\partial/\partial t)\, U(t, s) f = \big(-A(t) + L(t)\big)\, U(t, s) f \ \forall s, t \in R, f \in \mathscr{D}(R^6)$ $\big(f \in \mathscr{E}(R^6)$ *respectively*$\big)$, *and the map* $R^2 \ni (t, s) \to \big(-A(t) + L(t)\big)\, U(t, s) f \in X$ *is strongly continuous.*

To prove Theorem 3 we will use the following lemmas.

Lemma 2. *If assumptions* $(\mathrm{i})_1, (\mathrm{i})_2$ *and* $(\mathrm{j})_1$ *are true, then strongly continuous function* $R^2 \ni (t, s) \to U_0(t, s) \in \mathscr{B}(X)$ *verifies both condition* (i) *of* $(\mathrm{j})_4$ *and* (i)$'$ *of* $(\mathrm{j})_4'$.

Proof. We have already seen (property $(\beta)_{10}$) that if $f \in \mathscr{D}(R^6)$ then $U_0(t, s) f \in \mathscr{D}(R^6)$; furthermore if $f \in \mathscr{E}(R^6)$ then, because of $(\beta)_{11}$, also $U_0(t, s) f \in \mathscr{E}(R^6)$ and, because of $(\beta)_7$, map $(t, s) \to |x|\, U_4(t, s) f$ is strongly continuous.

Regarding the strong continuity of functions

$$(t, s) \to \frac{\partial}{\partial x_h}\, U_0(t, s) f\,, \qquad (t, s) \to |x|\, \frac{\partial}{\partial x_h}\, U_0(t, s) f \qquad\qquad h = 1, \ldots, 6\,,$$

$\big(f$ being an element of $\mathscr{D}(R^6)\big)$ it is an immediate consequence of formula (13), of $(\alpha)_9$, $(\beta)_1$, $(\beta)_5$, $(\beta)_6$, $(\beta)_7$ and of Lebesgue convergence theorem, if we consider that, because of $(\mathrm{j})_1$, real-valued functions $(x, t, s) \to (\partial\Phi_k/\partial x_h)(x, t, s)$ $(h, k = 1, \ldots, 6)$, exist and are continuous on all R^8.

Finally we can see, considering also $(\alpha)_3$ and $(\alpha)_6$, that $\forall s, t \in R^6$ it results

$$\| |x|\, U_0(t, s) f\| \leqq [\exp\int_{s\wedge t}^{s\vee t} (\bar\mu(\sigma) + \omega(\sigma))\, d\sigma][\| |x| f\| + \|f\|] \qquad \text{if } f \in \mathscr{E}(R^6);$$

$$(18) \qquad \sum_{h=1}^{6} \Big(\Big\|\frac{\partial}{\partial x_h} U_0(t, s) f\Big\| + \Big\| |x| \frac{\partial}{\partial x_h} U_0(t, s) f\Big\|\Big)$$

$$\leqq \Big(\exp\int_{s\wedge t}^{s\vee t} (\mu(\sigma) + \bar\mu(\sigma) + \omega(\sigma))\, d\sigma\Big)\, 12 \sum_{h=1}^{6} \Big(\Big\|\frac{\partial f}{\partial x_h}\Big\| + \Big\| |x| \frac{\partial f}{\partial x_h}\Big\|\Big) \qquad \text{if } f \in \mathscr{D}(R^6).$$

Lemma 3. *If assumptions* $(\mathrm{i})_1$-$(\mathrm{i})_7$-$(\mathrm{j})_1$ *and either* $(\mathrm{j})_2$-$(\mathrm{j})_4$ *or* $(\mathrm{j})_2'$-$(\mathrm{j})_4'$ *are true, if* $R^2 \ni (t, s) \to Z_0(t, s) \in \mathscr{B}(X)$ *is a strongly continuous function which satisfies either condition* (i) *or* (i)$'$ *respectively, then also*

$$(t, s) \to Z(t, s) = \int_s^t U_0(t, \sigma) L(\sigma) Z_0(\sigma, s)\, d\sigma \qquad \textit{is such a function}.$$

Proof. We suppose that assumptions $(\mathrm{i})_1$-$(\mathrm{i})_7$ and $(\mathrm{j})_1$-$(\mathrm{j})_4$ are true and that $Z_0(t, s)$ verifies condition (i) of $(\mathrm{j})_4$.

Then $\forall s, t \in R$ we have $Z(t, s) \in \mathscr{B}(X)$ and the map $(t, s) \to Z(t, s)$ from R^2 to $\mathscr{B}(X)$ is strongly continuous. Moreover, in virtue of $(\beta)_{10}$, $\forall s, \sigma, t \in R$ if $f \in \mathscr{D}(R^6)$, then also $U_0(t, \sigma) L(\sigma) Z_0(\sigma, s) f \in \mathscr{D}(R^6)$; furthermore Lemma 2 implies the strong continuity of the maps of R^3 into X

$$(t, \sigma, s) \to \frac{\partial}{\partial x_h} U_0(t, \sigma) L(\sigma) Z_0(\sigma, s) f,$$

$$(t, \sigma, s) \to |x| \frac{\partial}{\partial x_h} U_0(t, \sigma) L(\sigma) Z_0(\sigma, s) f \qquad h = 1, \ldots, 6.$$

From this the existence and the strong continuity on R^2 of these abstract integrals follows

$$\int_s^t \frac{\partial}{\partial x_h} U_0(t, \sigma) L(\sigma) Z_0(\sigma, s) f\, d\sigma, \quad \int_s^t |x| \frac{\partial}{\partial x_h} U_0(t, \sigma) L(\sigma) Z_0(\sigma, s) f\, d\sigma \qquad h = 1, \ldots, 6.$$

Considering this and the fact that $W^{1,1}(R^6)$ with the usual norm is a Banach space, we can state that $\forall f \in \mathscr{D}(R^6)$, $s, t \in R$ we have $Z(t, s) f \in \mathscr{D}(R^6)$, since

$$\frac{\partial}{\partial x_h} Z(t, s) f = \int_s^t \frac{\partial}{\partial x_h} U_0(t, \sigma) L(\sigma) Z_0(\sigma, s) f\, d\sigma \qquad h = 1, \ldots, 6.$$

The thesis of the case in question is therefore shown. The proof of the alternative case is similar.

Lemma 4. *If assumptions* $(\mathrm{i})_1$-$(\mathrm{i})_7$, $(\mathrm{j})_1$ *and either* $(\mathrm{j})_2$-$(\mathrm{j})_4$ or $(\mathrm{j})'_2$-$(\mathrm{j})'_4$ *are true, then* $\forall n \in N$ *the strongly continuous function* $R^2 \ni (t, s) \to U_n(t, s) \in \mathscr{B}(X)$ *satisfies either condition* (i) *or* (i)′ *respectively.*

Moreover in the first case the following inequality holds $\forall n \in N$, $s, t \in R$, $f \in \mathscr{D}(R^6)$

$$(19)\qquad \sum_{h=1}^{6} \Big(\Big\|\frac{\partial}{\partial x_h} U_n(t, s) f\Big\| + \Big\| |x| \frac{\partial}{\partial x_h} U_n(t, s) f\Big\|\Big)$$

$$\leqslant D_f \Big(\exp \int_{s\wedge t}^{s\vee t} (\mu(\sigma) + \bar{\mu}(\sigma) + \omega(\sigma))\, d\sigma\Big) \frac{1}{n!} \Big(\int_{s\wedge t}^{s\vee t} (b_0(\sigma) + \nu_0(\sigma) + 12\, l_1(\sigma))\, d\sigma\Big)^n,$$

if $$D_f = \|f\| + 12 \sum_{h=1}^{6} \Big(\Big\|\frac{\partial f}{\partial x_h}\Big\| + \Big\| |x| \frac{\partial f}{\partial x_h}\Big\|\Big);$$

whereas in the second case the following inequalities hold $\forall n \in N$, $s, t \in R$, $f \in \mathscr{E}(R^6)$

$$(20)\qquad \| |x|\, U_n(t, s) f\| \leqq \Big(\exp \int_{s\wedge t}^{s\vee t} (\bar{\mu}(\sigma) + \omega(\sigma))\, d\sigma\Big) \frac{\Big(\int_{s\wedge t}^{s\vee t} 3\lambda(\sigma)\, d\sigma\Big)^n}{n!} \big(\|f\| + \| |x| f\|\big),$$

$$(21)\qquad \sum_{h=1}^{6} \Big(\Big\|\frac{\partial}{\partial x_h} U_n(t, s) f\Big\| + \Big\| |x| \frac{\partial}{\partial x_h} U_n(t, s) f\Big\|\Big)$$

$$\leqq D'_f \Big(\exp \int_{s\wedge t}^{s\vee t} (\mu(\sigma) + \bar{\mu}(\sigma) + \omega(\sigma))\, d\sigma\, \frac{1}{n!} \Big(\int_{s\wedge t}^{s\vee t} (3\lambda(\sigma) + 12\, l_2(\sigma))\, d\sigma\Big)^n,$$

if $$D'_f = 2\|f\| + \| |x| f\| + 12 \sum_{h=1}^{6} \Big(\Big\|\frac{\partial f}{\partial x_h}\Big\| + \Big\| |x| \frac{\partial f}{\partial x_h}\Big\|\Big).$$

Proof. The first part of the thesis is an immediate consequence of the previous Lemmas 2 and 3.

To verify inequalities (19), (20) and (21), we first observe that we have, because of Lemma 1 and $(\beta)_2$ $\forall s, t \in R$

$$(22)\qquad \|U_n(t, s)\| \leqq \Big(\exp \int_{s\wedge t}^{s\vee t} \bar{\mu}(\sigma)\, d\sigma\Big) \frac{\Big(\int_{s\wedge t}^{s\vee t} (b_0(\sigma) + \nu_0(\sigma))\, d\sigma\Big)^n}{n!} \qquad n = 0, 1, 2, \ldots .$$

As a consequence of $(j)_3$, (18) and (22) we obtain, by induction, the following

$$(23)\qquad \sum_{h=1}^{6}\Big(\Big\|\frac{\partial}{\partial x_h}L(t)\,U_n(t,s)f\Big\| + \Big\|\,|x|\,\frac{\partial}{\partial x_h}L(t)\,U_n(t,s)f\Big\|$$

$$\leqslant l_1(t)\,D_f\Big(\exp\int_{s\wedge t}^{s\vee t}(\mu(\sigma)+\bar{\mu}(\sigma)+\omega(\sigma)\,\mathrm{d}\sigma\Big)\,\frac{1}{n!}\,\Big(\int_{s\wedge t}^{s\vee t}(b_0(\sigma)+\nu_0(\sigma)+12l_1(\sigma)\,\mathrm{d}\sigma\Big)^n$$

$$n = 0, 1, 2, \ldots .$$

From (18) and (23), (19) easily follows.

Similarly, inequalities (20) and (21) can readily be seen, by induction.

Proof of Theorem 3. Because of Theorem 1 and Lemma 1 we recognize that, if only assumptions $(i)_1$-$(i)_7$ are true, then family $\{U(t,s)\}$ possesses properties $(\delta)_1$-$(\delta)_5$.

The validity of $(\delta)_5$ can be shown by considering that $U(t,s)$ is also a solution of the integral equation

$$U(t,s) = U_0(t,s) + \int_s^t U(t,\sigma)L(\sigma)\,U_0(\sigma,s)\,\mathrm{d}\sigma\,.$$

We now suppose that also assumptions $(j)_1$-$(j)_4$ are true. Then it easily follows from Lemma 4 that $\forall f\in\mathscr{D}(R^6)$ also $U(t,s)f\in\mathscr{D}(R^6)$ and that the functions

$$(t,s)\to\frac{\partial}{\partial x_h}\,U(t,s)f\,,\qquad (t,s)\to|x|\,\frac{\partial}{\partial x_h}\,U(t,s)f\qquad\qquad h=1,\ldots,6$$

are strongly continuous.

This result, properties $(\delta)_1$-$(\delta)_5$, inequality (12) and the continuity of functions α_h imply the validity of $(\delta)_6$ in the case in question.

Similarly we can see that, in the other case, $(\delta)_6$ is true.

From Theorem 3 we immediately deduce the following

Corollary 1. *If assumptions* $(i)_1$-$(i)_7$ *and* $(j)_1$-$(j)_4$ *hold, then* $\forall f_0\in\mathscr{D}(R^6)$ *there exists one and only one continuous function* $R^+\ni t\to f(t)\in\mathscr{D}(R^6)\subseteq X$ *which is continuously differentiable and which is a solution of problem* (8).

This is the function $t\to U(t,0)f_0$.

Corollary 2. *If assumptions* $(i)_1$-$(i)_7$, $(j)_1$ *and* $(j)'_2$-$(j)'_4$ *hold, then* $\forall f_0\in\mathscr{E}(R^6)$ *there exists one and only one continuous function* $R^+\ni t\to f(t)\in\mathscr{E}(R^6)$ *which is continuously differentiable and which is a solution of problem* (8). *This is the function* $t\to U(t,0)f_0$.

Remark. It isn't difficult to find functions F, ν and k which satisfy all the assumptions in this section.

In particular the following propositions are obvious.

Proposition 1. If external force F and collision frequency ν satisfy assumptions $(\mathrm{i})_1$-$(\mathrm{i})_4$, $(\mathrm{j})_1$ and the following:

$(\mathrm{l})_1$ $\operatorname{div}_v F(x, t) = 0 \;\; \forall (x, t) \in R^7$,

$(\mathrm{l})_2$ $\forall h = 1, \ldots, 6$ and $\forall (x, t) \in R^7$ there exist the partial derivatives, $(\partial\nu/\partial x_h)(x, t)$ and the maps $(x, t) \to (\partial\nu/\partial x_h)(x, t)$ are continuous from R^7 to R,

$(\mathrm{l})_3$ there exists a continuous function $\bar\nu : R \to R^+$ such that $\forall h = 1, \ldots, 6$ and $\forall (x, t) \in R^7$ it results $|(\partial\nu/\partial x_h)(x, t)| \leqq \bar\nu(t)$,

then the family of bounded linear operators $\{U_\nu(t, s)\}$ defined by (17) has the following properties:

$(\varepsilon)_1$ $\|U_\nu(t, s)\| \leqq \exp \int_{s \wedge t}^{s \vee t} \nu_0(\sigma)\, d\sigma \;\; \forall s, t \in R$; $\|U_\nu(t, s)\| \leqq 1$ if $s \leqq t$.

$(\varepsilon)_2$ $U_\nu(t, t) = I \;\; \forall t \in R$.

$(\varepsilon)_3$ $U_\nu(t, \sigma)\, U_\nu(\sigma, s) = U_\nu(t, s) \;\; \forall \sigma, s, t \in R$.

$(\varepsilon)_4$ The map $(t, s) \to U_\nu(t, s)$ from R^2 to $\mathscr{B}(X)$ is strongly continuous.

$(\varepsilon)_5$ $(\partial/\partial s)\, U_\nu(t, s) f = U_\nu(t, s)(A(s) - B(s)) f \;\; \forall f \in \mathscr{D}(R^6)$, $s, t \in R$ and the map $R^2 \ni (t, s) \to U_\nu(t, s)(A(s) - B(s)) f \in X$ is strongly continuous.

$(\varepsilon)_6$ $(\partial/\partial t)\, U_\nu(t, s) f = (-A(t) + B(t))\, U_\nu(t, s) f \;\; \forall f \in \mathscr{E}(R^6)$, $s, t \in R$ and the map $R^2 \ni (t, s) \to (-A(t) + B(t))\, U_\nu(t, s) f \in X$ is strongly continuous.

Proposition 2. We suppose that the external force, the collision frequency and the collision nucleus satisfy assumptions $(\mathrm{j})_3$-$(\mathrm{i})_7$, $(\mathrm{l})_1$-$(\mathrm{l})_3$ and the following:

$(\mathrm{l})_4$ $\forall (r, v, v', t) \in R^{10}$ there exist the first order partial derivatives $(\partial k/\partial x_h)(r, v, v', t)$ $(h = 1, \ldots, 6, (x_1, \ldots, x_6) = (r, v))$ and $\forall t \in R$, $h = 1, \ldots, 6$ the functions

$$R^9 \ni (r, v, v') \to \frac{\partial k}{\partial x_h}(r, v, v', t) \in R \quad \text{are measurable and bounded.}$$

$(\mathrm{l})_5$ $\forall (r, v', t) \in R^7$ the functions

$$v \to |v|\, k(r, v, v', t)\,, \qquad v \to \frac{\partial k}{\partial x_h}(r, v, v', t) \qquad v \to |v|\, \frac{\partial k}{\partial x_h}(r, v, v', t)$$

$$h = 1, \ldots, 6\,,$$

are Lebesgue integrable on R^3 and there exist three continuous functions $m_i\colon R \to R^+$ $(i = 1, 2, 3)$ so that $\forall (r, v', t) \in R^7$ it results

$$\int_{R^6} |v|\, k(r, v, v', t)\, dv \leqq m_1(t)\,,$$

$$\int_{R^6} \left|\frac{\partial k}{\partial x_h}(r, v, v', t)\right| dv \leqq m_2(t),$$

$$\int_{R^6} |v| \left|\frac{\partial k}{\partial x_h}(r, v, v', t)\right| dv \leqq m_3(t) \qquad h = 1, \dots, 6\,.$$

$(1)_6$ Arbitrarily fixed $\tau \in R$ we have for a.e. $(r, v') \in R^6$

$$\lim_{t\to\tau} \int_{R^6} |v|\, |k(r, v, v', t) - k(r, v, v', \tau)|\, dv = 0\,,$$

$$\lim_{t\to\tau} \int_{R^6} \left|\frac{\partial k}{\partial x_h}(r, v, v', t) - \frac{\partial k}{\partial x_h}(r, v, v', \tau)\right| dv = 0\,,$$

$$\lim_{t\to\tau} \int_{R^6} |v| \left|\frac{\partial k}{\partial x_h}(r, v, v', t) - \frac{\partial k}{\partial x_h}(r, v, v', \tau)\right| dv = 0 \qquad h = 1, \dots, 6\,.$$

Then the family of bounded linear operators $\{L(t)\}$ satisfies conditions $(\mathrm{j})'_2$, $(\mathrm{j})'_3$ and $(\mathrm{j})'_4$.

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Riassunto

Viene studiato il problema ai valori iniziali per l'equazione lineare di Maxwell-Boltzmann, nell'ipotesi che la forza esterna F, la frequenza ν e il nucleo k di collisione dipendano dalle variabili di stato e dal tempo. Usando la teoria degli operatori di evoluzione, si dimostra dapprima un teorema di esistenza e unicità di una soluzione globale, debole, non negativa; si riconosce infine che, per convenienti F, ν e k, la soluzione suddetta è anche soluzione forte.

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