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On 2-codimensional co-isotropic submanifold with parallel complementary distribution in a neutral pseudo-riemannian manifold (**)

Introduction

Let $\tilde{M}(U, \tilde{\Omega}, \tilde{g})$ be a neutral pseudo-Riemannian manifold of dimension $2m$, whose tensor fields $(U, \tilde{\Omega}, \tilde{g})$ are the para-complex operator [5], the canonical almost symplectic form and the para-Hermitian metric tensor.

In [1] we have studied some properties of 2-codimensional *co-isotropic* [7] submanifold M of $\tilde{M}$, and showed that M is always a CR-submanifold [2] whose vertical distribution $D^{\perp}$ is self-orthogonal [7]. Denoting by $CD^{\perp}$, the self-orthogonal complementary distribution of $D^{\perp}$, we study, in the present paper, the case when $CD^{\perp}$ is *parallel* in D, that is $\nabla CD^{\perp} \subset D^{\perp} \oplus CD^{\perp}$ (∇: operator of covariant differentiation on M).

The following properties are proved:

(i) The simple unit form φ which corresponds to $D^{\perp}_p (p \in M)$ is exterior recurrent and any vector field Z of the horizontal distribution D is a conformal infinitesimal transformation of φ.

(ii) D is involutive and the leaves M_I of D are invariant submanifold of $\tilde{M}$, and such that the proper immersion $x \colon M_I \to \tilde{M}$ is *pseudo-minimal* (in the sense of R. Rosca $[6]_1$).

1 – Let $\tilde{M}(U, \tilde{\Omega}, \tilde{g})$ be a neutral C^{∞}-pseudo-Riemannian manifold of dimension $2m$. The structure tensor are, the *para-complex* operator U of P. Libermann [5], the para-Hermitian metric tensor $\tilde{g}$ and the canonical *almost symplectic* 2-form $\tilde{\Omega}$, exchangeable with $\tilde{g}$.

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In [1] we have studied some properties of the improper immersion $x\colon M \to \tilde{M}(U, \tilde{\Omega}, \tilde{g})$, where M is a 2-codimensional co-isotropic submanifold of $\tilde{M}$.

We have shoved that M is always a CR-submanifold of $\tilde{M}$, whose *vertical distribution* $D^{\perp}$ is self-orthogonal [7].

If $T_{\tilde{p}}(\tilde{M})$ is the tangent space to $\tilde{M}$ at $\forall \tilde{p} \in \tilde{M}$, then one has $T_{\tilde{p}}(\tilde{M})|_M = D_p \oplus D_p^{\perp} \oplus CD_p^{\perp}$, where D_p and $CD_p^{\perp}$ are the *horizontal distribution* and the *complementary self-orthogonal* distribution of $D_p^{\perp}$ at $\forall p \in M$ (we denote the induced elements by the mapping $x\colon M \to \tilde{M}$ by suppressing $\sim$).

Let $W = \{h_a, h_{a^*}; a = 1, \dots, m; a^* = a + m\}$ be a local field of Witt vectorial basis. Since W is normed, one has [**6**]$_2$

$$(1.1) \qquad \langle h_a, h_{b^*} \rangle = \delta_{ab} .$$

The operator U defines an involutive automorphism on $\tilde{M}(U^2 = +1)$ and one has

$$(1.2) \qquad U h_a = h_a \qquad U h_{a^*} = -h_{a^*} .$$

If $W^* = \{\tilde{\omega}^A, A \in \{a, a^*\}\}$ is the cobasis of W and $\tilde{\omega}_B^A = \tilde{l}_{BC}^A \tilde{\omega}^C (\tilde{l}_{BC}^A \in C^\infty(\tilde{M}))$ the connection formes associated with W^*, then the line element $\mathrm{d}\tilde{p}$ ($\mathrm{d}\tilde{p}$ is a canonical vector 1-form on the tangent bundle $T(\tilde{M})$) and the connection equations are given by

$$(1.3) \qquad \mathrm{d}\tilde{p} = \tilde{\omega}^A \otimes h_A \qquad\qquad (1.4) \qquad \tilde{\nabla} h_A = \tilde{\omega}_A^B \otimes hB$$

($\tilde{\nabla}$: covariant differentiation operator on $\tilde{M}$) respectively.

If $\tilde{\Omega}_B^A$ are the curvature 2-forms on $\tilde{M}$, then the structure equations (E. Cartan) are expressed by

$$(1.5) \qquad \mathrm{d}\tilde{\omega}^A = \tilde{\omega}^B \wedge \tilde{\omega}_B^A \qquad\qquad (1.6) \qquad \mathrm{d}\tilde{\omega}_B^A = \tilde{\Omega}_B^A + \tilde{\omega}_B^C \wedge \tilde{\omega}_C^A$$

(the connection $\tilde{\nabla}$ is torsion-less).

Further by (1.1) and (1.3) one has

$$(1.7) \qquad \tilde{g} = \langle \mathrm{d}\tilde{p}, \mathrm{d}\tilde{p} \rangle = 2 \sum_a \tilde{\omega}^a \otimes \tilde{\omega}^{a^*}$$

and $\tilde{g}$ is exchangeable with the almost symplectic form

$$(1.8) \qquad \tilde{\Omega} = \sum_a \tilde{\omega}^a \wedge \tilde{\omega}^{a^*} .$$

Finally by (1.1) and (1.4) one finds [**6**]$_2$

$$(1.9) \qquad \tilde{\omega}_b^a + \tilde{\omega}_{a^*}^{b^*} = 0 \qquad \tilde{\omega}_b^{a^*} + \tilde{\omega}_a^{b^*} = 0 \qquad \tilde{\omega}_{b^*}^a + \tilde{\omega}_{a^*}^b = 0 .$$

2 – Without loss of generality, a 2-codimensional co-isotopic submanifold M of $\tilde{M}$ may be defined by [1]

$$\omega^{2m} = 0 \qquad \omega^{2m-1} = 0 \,. \tag{2.1}$$

In this case the line element dp of M is expressed by

$$dp = \omega^i \otimes h_i + \omega^{i^*} \otimes h_{i^*} + \omega^r \otimes h_r \tag{2.2}$$

where we agree with the following range of indices $i = 1, \dots, m-2$, $i^* = i + m$, $r = m-1, n$.

Then according to [1] the vertical distribution $D_p^\perp$ and its self-orthogonal complementary on $CD_p^\perp$ are defined by $\{h_r\}$ and $\{h_{s^*}\}$ respectively.

We shall suppose in this paper that $CD_p^\perp$ is parallel in D.

Then according to [3] the covariant derivative of any vector field of $CD_p^\perp$ has no components in D. If ∇ is the restriction of $\tilde{\nabla}$ on M, one may write

$$\nabla CD^\perp \subset D^\perp \oplus CD^\perp \,. \tag{2.3}$$

Since $CD^\perp = \{h_{s^*}\}$ one finds by (2.3) and (1.4)

$$\omega_i^r = 0 \qquad \omega_{i^*}^r = 0 \,. \tag{2.4}$$

Denote now by

$$\varphi = \omega^{m-1} \wedge \omega m \tag{2.5}$$

the *simple unit form* which corresponds to the vertical distribution $D_p^\perp$. Making use of (1.5) and (2.4) and taking the exterior derivative of φ one gets

$$d\varphi = -(\Sigma_r \omega_r^r) \wedge \varphi. \tag{2.6}$$

The above equation shows that φ *is exterior recurrent* [4] and has $-\Sigma_r \omega_r^r$ as recurrence 1-form.

Put $u = -\Sigma_r \omega_r^r$ and let Z be any vector field of D_p.

Since $i_Z \varphi = 0$ (i_Z: interior product by Z) one readly finds by (2.6) that the Lie derivative $(L_Z = d \circ i_Z + i_Z \circ d)$ in the direction Z is expressed by

$$L_Z \varphi = u(Z) \varphi \,. \tag{2.7}$$

Hence any vector field $Z \in D_p$ is a *conformal infinitesimal transformation* of φ.

Further let $\forall Z,\ Z' \in D_p$. Then by (1.4) and (2.4) one readly finds

$$[Z, Z'] \in D_p \qquad ([\ ]\text{: Lie bracket}) . \tag{2.8}$$

Hence the horizontal distribution D is *involutive.*

Denote by M_I the leaves of D and by $T_p(M_I)$ the tangent space to M_I at $\forall p \in M_I$.

Since $D_p = \{h_i, h_{i^*}\} = T_p(M_I)$, one has $UT_p(M_I) = T_p(M_I)$ and M_I is an *invariant* [**9**] submanifold of $\tilde{M}$.

Next if Z and Z' are any tangent vector fields of M_I then the Gauss equation associated wits $x: M_I \to \tilde{M}(U, \tilde{\Omega}, \tilde{g})$ is as is known given by

$$\tilde{\nabla}_{Z'} Z = \tilde{\nabla}_{Z'} Z + B(Z, Z') \tag{2.9}$$

where B (the normal part of $\tilde{\nabla}_{Z'} Z$) is the second fundamental normal tensor of M_I.

Putting g_I for the metric tensor of M_I, one has

$$\operatorname{tr} g_I B = (\dim M) H \tag{2.10}$$

where H is the *mean curvature vector* of M_I.

Referring to (1.7) and [**6**]$_1$ we have in the case under discussion

$$H = \frac{1}{2(m-2)} \Sigma_i B(h_i, h_{i^*}) . \tag{2.11}$$

Now using (2.4), we get after a short calculation

$$H = \frac{1}{2(m-2)} \Sigma \, (l^{r^*}_{ii^*} + l^{r^*}_{i^*i}) \, h_{r^*} \tag{2.12}$$

and by (1.1) one gets instantly $\langle H, H \rangle = 0$.

Hence H is a null vector field and, so according to a definition of R. Rosca [**6**]$_1$, the submanifold M_I under consideration is *pseudo-minimal.*

Theorem. *Let $x: M \to \tilde{M}(U, \tilde{\Omega}, \tilde{g})$ be the improper immersion of a 2-codimensional co-isotropic submanifold M in a neutral pseudo-Riemannian manifold $\tilde{M}$ and let D, $D^\perp$ and $CD^\perp$ be the horizontal, the vertical and the complementary self-orthogonal distribution of $D^\perp$, respectively.*

If $CD^\perp$ is parallel in D, then one has the following properties:

(i) *The simple unit form φ which corresponds to $D^\perp_p (\forall p \in M)$ is exterior recurrent, and any vector field $Z \in D_p$ is a conformal infinitesimal transformation of φ.*

(ii) *D is involutive and the leaves M_I of D are invariant submanifolds of $\tilde{M}$ and such that the proper immersion $x: M_I \to \tilde{M}$ is pseudo-minimal.*

References

[1] F. Amato, *CR-sous-variétés co-isotropes incluses dans une C^∞-variété pseudo riemannienne neutre*, Boll. Un. Mat. Ital. (6) **4-A** (1985), 433-440.

[2] A. Bejancu, *CR-submanifold of a Kaehler manifold* (I), Proc. Amer. Math. Soc. **69** (1978), 135-142.

[3] B. Y. Chen, *Geometry of submanifolds*, M. Dekker, New York, 1973.

[4] D. K. Datta, *Exterior recurrent forms on a manifold*, Tensor N. S. **36** (1982), 115-120.

[5] P. Libermann, *Sur le problème d'équivalence de certaines structures infinitésimales*, Ann. Mat. **36** (1954), 27-119.

[6] R. Rosca: $[\bullet]_1$ *Sous-variétés pseudo-minimales et minimales d'une variété pseudo-riemannienne structurée par une connexion spin-euclidienne*, C.R. Acad. Sc. Paris Sér. A **290** (1980), 331-333; $[\bullet]_2$ *Parakählerian manifolds carrying a pair of concurrent self-orthogonal vector fields*, Abh. Math. Sem. Univ. Hamburg (1976), 207-217.

[7] J. M. Souriau, *Structures des systèmes dynamiques*, Dunod, Paris, 1970.

[8] K. Yano and M. Kon, *Anti-invariant submanifolds*, M. Dekker, New York, 1976.

Riassunto

Sia M una sottovarietà co-isotropa a 2 codimensioni di una varietà neutra $\tilde{M}(U, \tilde{\Omega}, \tilde{g})$ a $2m$ dimensioni. Si considerano su $\tilde{M}$ tre distribuzioni, e cioè: una distribuzione invariante D, una distribuzione isotropa $D^\perp$ e la distribuzione isotropa $CD^\perp$ complementare di $D^\perp$ tale che in ogni punto $\bar{p}$ di $\tilde{M}$ lo spazio tangente $T_{\bar{p}}(\tilde{M})$ è definito da $D_{\bar{p}} \oplus D_{\bar{p}} \oplus CD_{\bar{p}}$. Si suppone inoltre che $CD^\perp$ sia parallelo in D, ossia $\nabla CD^\perp \subset D^\perp \oplus CD^\perp$, dove ∇ è l'operatore di differenziazione covariente su M. Dalle precedenti condizioni si deduce: (a) *Ogni campo vettoriale $Z \in D$ definisce una trasformazione infinitesima conforme della forma semplice φ che corrisponde a $D^\perp$.* (b) *La CR-sottovarietà M è fogliettata e le foglie M_I di D sono pseudo-minimali.*

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