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On semi-prime and reduced rings

Dedicated to Prof. Carl Faith for his sixtieth birthday

Introduction

Throughout, A represents an associative ring with identity and A-modules are unital. J, Z, Y will denote respectively the Jacobson radical, the left singular ideal and the right singular ideal of A. As usual: (1) a left (right) ideal of A is called *reduced* if it contains no non-zero nilpotent element (it is evident that if A is reduced, then A is semi-prime but the converse is not true); (2) an ideal of A always means a two-sided ideal; (3) A is called a *right V-ring* if every simple right A-module is injective; (4) an element a of A is called *von Neumann regular* if $a \in aAa$; A is therefore von Neumann regular iff every element is von Neumann regular; (5) a right A-module M is *p-injective* iff for any $a \in A$, any right A-homomorphism of aA into M extends to one of A into M. A is von Neumann regular iff every left (right) A-module is p-injective. Note that a flat A-module needs not be p-injective and the converse is not true either. However, for any p-injective right ideal I of A, A/I is a flat right A-module. If A is quasi-Frobeniusean, then a right ideal is injective iff it is p-injective iff it is flat. If $Z(M)$ denotes the singular submodule of a left A-module M, then M is singular (resp. non-singular) iff $Z(M) = M$ (resp. $Z(M) = 0$). Thus A is left non-singular iff $Z = 0$. Note that A is von Neumann regular iff every cyclic singular left A-module is flat $[9]_2$.

The purpose of this note is to consider various generalizations of reduced rings through N-injectivity which is now introduced.

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1 – Definitions. A right A-module M is called *N-injective* if, for any nilpotent element b of A, any right A-homomorphism of bA into M extends to one of A into M.

It is obvious that if A is reduced or von Neumann regular, then every right (left) A-module is N-injective. A is called a *right N-injective ring* if A_A is N-injective.

Now call A a *right* SNI *ring* if every simple right A-module is N-injective. Right SNI rings generalize effectively von Neumann regular rings, right V-rings and reduced rings. A simple N-injective module needs not be p-injective (otherwise, any commutation non-singular ring would be von Neumann regular!).

Note that if I is a maximal left ideal of A which is an ideal, then A/I_A is flat iff $_AA/I$ is injective iff $_AA/I$ is p-injective ([9]$_5$, Lemma 1).

Our first result shows that right SNI rings constitute a class of rings strictly between reduced rings and semi-prime rings.

Proposition 1. *If A is a right* SNI *ring, then:* (1) $Z=0$; (2) *J contains no non-zero nilpotent element* (*conseguently, A is semi-prime*).

Proof. (1) If $Z\neq 0$, then by [9]$_6$ (Lemma 7), there exists $0\neq z\in Z$ such that $z^2=0$. First suppose that $Z+r(z)\neq A$. If R is a maximal right ideal containing $Z+r(z)$, then A/R_A is N-injective and if $f\colon zA\to A/R$ is the right A-homomorphism defined by $f(za)=a+R$ for all $a\in A$, there exists $c\in A$ such that $1+R=f(z)=cz+R$, whence $1-cz\in R$, yielding $1\in R$, which contradicts $A\neq R$. Now suppose that $Z+r(z)=A$. Then $1=w+u$, $w\in Z$, $u\in r(z)$ which implies $z=zw$, whence $Az\cap l(w)=0$. Since $w\in Z$, $z=0$ which contradicts the hypothesis that $z\neq 0$. This proves (1).

(2) If $0\neq u\in J$ such that $u^2=0$, then $J+r(u)\neq A$ leads to a contradiction as in (1). Now suppose that $J+r(u)=A$. Then $u=uv$ for some $v\in J$ and since $(1-v)d=1$ for some $d\in A$, then $u(1-v)=0$ implies $u=0$, a contradiction! Since any left (or right) nilideal is contained in the Jacobson radical, then A must be semi-prime.

Our first corollary follows from the fact that any reduced principal right ideal of a right self-injective ring is generated by an idempotent.

Corollary 1.1. *If A is a right* SNI *ring which is either right or left self-injective, then A is von Neumann regular.*

Corollary 1.2. *If every complement left ideal of A is an ideal, then A is reduced iff A is right* SNI. *In that case, every complement left ideal of A is an annihilator* (cfr. [9]$_1$, Lemma 1).

Applying $[9]_3$ (Proposition 6), we get

Corollary 1.3. *If A is a prime right* SNI *ring, then either $J=0$ or A is an integral domain.*

Corollary 1.4. *If A is right* SNI *which is either right p-injective or π-regular, then $J=Y=0$* (cfr. $[9]_7$, Proposition 3).

Left CM-rings, which generalize left PCI rings $[5]_1$, left Ore domains, left duo rings and semi-simple Artinian rings, are studied in $[9]_4$. Applying $[9]_4$, (Lemma 1.1) to Proposition 1(1), we get

Remark 1. The following conditions are equivalent for a left CM-ring A: (a) A is either semi-simple Artinian or reduced; (b) A is right SNI.

Remark 2. If every essential right ideal of A is an ideal, then A is von Neumann regular iff every factor ring of A is right SNI.

Remark 3. A direct sum of right A-modules is N-injective iff each direct summand is N-injective.

Remark 4. The following conditions are equivalent: (a) every nilpotent element of A is von Neumann regular; (b) every right A-module is N-injective; (c) every left A-module is N-injective.

Applying the theorem in [1] to Proposition 1(2), we get

Remark 5. The following conditions are equivalent: (a) A is a commutative reduced ring; (b) A is a right SNI ring such that there exist a non-negative integer k and a positive integer m satisfying $a^m ba^k ba^m = ba^{2m+k} b$ for all $a, b \in A$.

2 – The following N-injective analog of the injective characterization of hereditary rings also holds (cfr. [4], p. 14), yielding a particular class of right non-singular rings.

Proposition 2. *The following conditions are equivalent:*

(1) *Every principal right ideal of A generated by a nilpotent element is a projective right A-module.*

(2) *Every quotient of a N-injective right A-module is N-injective.*

(3) *Every quotient module of an injective right A-module is N-injective.*

(4) *If $\hat{M}$ is the injective hull of M_A, then $\hat{M}/M_A$ is N-injective.*

(5) *The sum of any two N-injective submodules of any right A-module is N-injective.*

(6) *The sum of any two injective submodules of any right A-module is N-injective.*

Rings satisfying the equivalent conditions of Proposition 2 provide a nice generalization of von Neumann regular rings, right hereditary rings and reduced rings.

Corollary 2.1. *The following conditions are equivalent:*

(1) *A is finite direct sum of division rings.*

(2) *A is a right duo ring such that the sum of any two N-injective right A-modules is injective.*

Following $[\mathbf{9}]_8$, A is called a *right* IQC *ring* if, for any essential right ideal I such that there exist a non-zero complement right ideal of A isomorphic to I/K for some right subideal K, then any right A-homomorphism of I into A extends to an endomorphism of A_A.

Combining $[\mathbf{9}]_7$ (Theorem 5), $[\mathbf{9}]_8$ (Corollary 1.2) with Propositions 1 and 2, we get a few characteristic properties of self-injective regular rings.

Proposition 3. *The following conditions are equivalent for a right* IQC *ring A:*

(1) *A is right self-injective regular.*

(2) *Every quotient of an injective right A-module is N-injective.*

(3) *Every cyclic singular right A-module is N-injective.*

(4) *A is left* SNI.

The proof of $[\mathbf{9}]_8$ (Theorem 2) yields the following

Remark 6. If every N-injective right A-module is quasi-injective, then A is right Noetherian such that every N-injective right A-module is injective (apply Remark 3).

Right N-injective rings generalize right p-injective rings and reduced rings. (Note that a reduced right p-injective ring is strongly regular). Quasi-Frobeniusean rings are now characterized in terms of N-injective rings (this completes $[9]_8$, Theorem 4).

Proposition 4. *The following conditions are equivalent:*

(1) *A is quasi-Frobeniusean.*

(2) *A is a left* IQC *left Noetherian left and right N-injective ring.*

(3) *A is a left p-injective left Noetherian right N-injective ring.*

(4) *A is a left Artinian, left and right N-injective ring.*

Proof. Obviously, (1) implies (2) and (3).

Assume (2). Then A is left continuous ($[9]_8$, Theorem 1), which implies that A/J is semi-simple Artinian. Since $J=Z$ is nilpotent, then A is semi-primary which yields A left Artinian, whence (2) implies (4).

Since a left p-injective ring satisfying the maximum condition on left annihilators is semi-primary, then (3) implies (4).

Now assume (4). Let $U=Au$, $u \in A$, be a minimal left ideal which is not generated by an idempotent. Then $U^2=0$. If $v \in l(r(u))=l(r(U))$, define a right A-homomorphism $g\colon uA \to A$ by $g(ua)=va$ for all $a \in A$. Then g is well-defined and since $u^2=0$, there exists $y \in A$ such that $v=g(u)=yu \in Au$. This proves that $U=l(r(U))$. Therefore any minimal left ideal must be a left annihilator. Similarly, any minimal right ideal of A is a right annihilator. By [8] (Proposition 1), (4) implies (1).

Remark 7. If each prime factor ring of A is right p-injective, then A is right Artinian iff A is right Noetherian.

A is called a *right Kasch ring* if every maximal right ideal is a right annihilator $[5]_2$. If A is right perfect, then any p-injective right ideal is a direct summand of A_A. The next proposition then follows from Proposition 1 (2) and Proposition 2 (cfr. also [7], p. 332).

Proposition 5. *The following conditions are equivalent:*

(1) *A is semi-simple Artinian.*

(2) *A is a right* SNI *right Kasch ring.*

(3) *A is a right Kasch ring such that the sum of any two injective right A-modules is N-injective.*

(4) *A satisfies the maximum condition on right annihilators and every ideal of A is right p-injective.*

(5) *A is a right p-injective right* SNI *ring satisfying the maximum condition on right annihilators.*

(6) *A is a right* SNI, *right or left perfect ring.*

Remark 8. A right p-injective left pseudo-coherent left Kasch ring is left Artinian iff it satisfies the minimum condition on left annihilators.

Proposition 6. *The following conditions are equivalent:*

(1) *A is either a right p-injective local ring whose Jacobson radical is a nilideal or strongly regular.*

(2) *Every non-nil left ideal of A is p-injective right A-module.*

Proof. If A is a right p-injective local ring whose Jacobson radical is nil, then the only non-nil left ideal is A. Therefore (1) implies (2).

Assume (2). First suppose that A contains a maximal left ideal K which is a nilideal. Then $K \subseteq J$ which implies $J = K$ is the unique maximal left (and hence right) ideal of A. Therefore A is a local ring such that J is nil and also A is right p-injective. Now suppose that every maximal left ideal is non-nil. Then every maximal left ideal of A is right p-injective, yielding $Ab + l(b) = A$ for any $B \in A$. In that case, A is strongly regular. Thus (2) implies (1).

3 – At this point, let us introduce strongly reduced one-sided ideals.

Def. A right ideal I of A is called a *strongly reduced right ideal* if, for any $a \in A$, aI is a reduced right ideal of A. *Strongly reduced left ideals* are similarly defined. It is obvious that *reduced ideals* are strongly reduced right and left ideals.

Proposition 7. *If A is semi-prime, the following three sets coincide*: (a) *the sum of all strongly reduced right ideals of A*; (b) *the sum of all strongly reduced left ideals of A*; (c) *the sum of all reduced ideals of A. This is precisely the unique maximal reduced ideal of A.*

Proof. Let S denote the sum of all strongly reduced right ideals of A, $W = l(r(S))$. We prove that W is a reduced ideal of A which will yield $W = S$. Suppose that W is not reduced. Then there exists $0 \neq w \in W$ such that $w^2 = 0$. If $Sw = 0$, since W is an ideal of A, then $(Aw)^2 = (AwA)w \subseteq Ww \subseteq Wr(S) = 0$, which contradicts A semi-prime. If $Sw \neq 0$, there exists a non-zero strongly reduced right ideal R of A such that $Rw \neq 0$. Since R is a reduced right ideal, then $wR \neq 0$. If $d \in R$ such that $dw \neq 0$, then wd is a non-nilpotent element of wR, whence $wdw \neq 0$. But $(wdw)^2 = wdw^2dw = 0$ which contradicts wR reduced. This proves that W is a reduced ideal of A which is therefore a strongly reduced right ideal. It follows that $W = S$. It is clear that the same proof shows that both the sum of all strongly reduced left ideals of A and the sum of all reduced ideals of A coincide with S. Therefore S is the unique maximal reduced ideal of A.

Recall that A is a *Baer ring* if every left annihilator ideal (equivalently, every right annihilator ideal) is generated by an idempotent.

Corollary 7.1. *Let A be a semi-prime Baer ring, S the sum of all strongly reduced right ideals of A. Then $A = S \oplus T$, where T is an ideal of A containing all the nilpotent elements of A and any non-zero ideal of T must have a non-zero nilpotent element.*

Applying $[5]_1$ (Lemma 20.27), we get

Corollary 7.2. *If A is a left Noetherian right semi-hereditary right* SNI *ring, then $A = S \oplus T$, where S is the unique maximal reduced ideal of A and T is the minimal direct summand of A containing all the nilpotent elements.*

I is called a *strongly regular ideal* of A if I is a reduced ideal such that for any $b \in I$, there exists $c \in I$ such that $b = bcb$. If either A is right p-injective or every simple right A-module is flat, then any reduced principal right ideal of A is generated by an idempotent (cfr. $[9]_5$, p. 447).

Corollary 7.3. *Let A be a right* SNI *ring satisfying any one of the following conditions*: (a) *A is right p-injective*; (b) *A is left p-injective*; (c) *every simple right A-module is flat*; (d) *every simple left A-module is flat. Then S, the sum of all strongly reduced right ideals of A* (*which is the sum of all strongly reduced left ideals of A*), *is the unique maximal strongly regular ideal of A and $S = l(r(S))$.*

We now give an additional information on a decomposition of Birkenmeier ([3], Corollary 14).

Remark 9. If A is a semi-prime ring whose complement right ideals are direct summands of A_A, then $A = S \oplus T$, where S is the sum of all strongly reduced right ideals of A and is therefore the unique maximal reduced ideal of A, T is the minimal direct summand containing the nilpotent elements of A.

We now turn to a result motivated by $[9]_4$ (Remark 2(c)) and [2].

Proposition 8. *Suppose that the essential left ideals of A coincide with the essential right ideals of A. The following are then equivalent* : (a) *every singular left A-module is injective*; (b) *every singular right A-module is injective. In that case, A is regular left and right hereditary ring whose socle is left and right essential.*

Proof. Suppose that (a) holds. For any $z \in Z$, ${}_AAz$, being injective, is a direct summand of ${}_AA$ which implies $z = 0$. Therefore $Z = 0$. Now assume that A is not semi-prime. If T is a non-zero ideal of A such that $T^2 = 0$, K a relative complement of T_A in A_A, then $E = T \oplus K$ is an essential right ideal which is therefore left essential. Now $ET = 0$ implies $T \subseteq Z = 0$, a contradiction! This proves that A must be semi-prime. In as much as A is a semi-prime ring whose simple left modules are either injective or projective, then we know that A is fully left idempotent which implies A regular by $[9]_2$ (Proposition 9), whence every singular right A-module is injective ([2], Proposition 2.9). Thus the equivalence of (a) and (b) is established. We know that if, for any left A-module M with injective hull H, ${}_AH/M$ is injective, then A is left hereditary (cfr. $[9]_2$, Proposition 4). Since ${}_AH/M$ is singular, then (a) implies that A is left hereditary. The last part follows from [2] (Proposition 2.9).

Our last remark follows from [2] (Theorem 3.1) and the proof of Proposition 8.

Remark 10. Let A be a left non-singular ring whose simple left modules are flat such that either A is left CM or every essential right ideal of A is left essential. Then A is von Neumann regular.

References

[1] A. H. AL-MOAJIL, *A note on rings with no nil right ideals*, Arabian J. Sci. Eng. 8 (1983), 145-146.

[2] G. BACCELLA, *Generalized V-rings and von Neumann regular rings*, Rend. Sem. Mat. Univ. Padova 72 (1984), 117-133.

[3] G. F. BIRKENMEIER, *Baer rings and quasi-continuous rings have a* MDSN, Pac. J. Math. 97 (1981), 283-292.

[4] H. CARTAN and S. EILENBERG, *Homological Algebra*, Princeton, 1956.

[5] C. FAITH: [•]$_1$ *Algebra* II: *Ring Theory*, Springer-Verlag **191** (1976); [•]$_2$ *Injective modules and injective quotient rings*, Lecture Notes n. 72, Dekker, 1982.

[6] K. R. GOODEARL, *Von Neumann regular rings*, Pitman, London, 1979.

[7] F. KASCH, *Modules and rings*, London Math. Soc. Monograph 17 (translated by D. A. R. Wallace), 1982.

[8] H. H. STORRER, *A note on quasi-Frobenius rings and ring epimorphisms*, Canad. Math. Bull. **12** (1960), 287-292.

[9] R. YUE CHI MING: [•]$_1$ *On annihilator ideals*, Math. J. Okayama Univ. **19** (1976), 51-53; [•]$_2$ *On generalizations of V-rings and regular rings*, Math. J. Okayama Univ. **20** (1978), 123-129; [•]$_3$ *On V-rings and prime rings*, J. Algebra **62** (1980), 13-20; [•]$_4$ *On regular rings and continuous rings* (II), Kyungpook Math. J. **21** (1981), 171-178; [•]$_5$ *On regular rings and Artinian rings*, Riv. Mat. Univ. Parma (4) 8 (1982), 443-452; [•]$_6$ *On quasi-injectivity and von Neumann regularity*, Monatsh. Math. **95** (1983),25-32; [•]$_7$ *On von Neumann regular rings* (VIII), J. Korean Math. Soc. **19** (1983), 97-104; [•]$_8$ *On von Neumann regular rings* (IX), Riv. Mat. Univ. Parma (4) 9 (1983), 403-410.

Summary

In this note, N-injective modules are introduced to study various generalizations of reduced rings : (a) *rings whose simple right modules are N-injective form an interesting class of rings intermediate between reduced rings and semi-prime rings*; (b) *rings whose quotient modules of injective right modules are N-injective generalize effectively right hereditary rings and reduced rings. Self-injective regular rings and quasi-Frobeniusean rings are characterized. If A is a semi-prime ring, the sum of all strongly reduced right ideals of A is equal to the sum of all strongly reduced left ideals of A* (*which is proved to be the unique maximal reduced two-sided ideal of A*).

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