Riv. Mat. Univ. Parma (4) **12** (1986), 187-194

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Almost contact submersions and $K_{i\xi}$-curvatures (**)

1 - Preliminaries

Let (M, ϕ, ξ, η, g) and $(M', \phi', \xi', \eta', g')$ be almost contact metric manifolds of dimension $2m+1$ and $2m'+1$ respectively. We shall represent by $\overline{\nabla}$ and ∇' the Riemannian connections of M and M' respectively.

A Riemannian submersion $f: M \to M'$ is called an *almost contact metric submersion* if $\phi' \cdot f_* = f_* \cdot \phi$. It follows that $f_* \xi = \pm \xi'$. We shall suppose that $f_* \xi = \xi'$.

For an almost contact submersion, we denote the vertical and horizontal distributions in $T(M)$ by $V(M)$ and $H(M)$ and the orthogonal projections mappings by $\mathscr{V}$ and $\mathscr{H}$ respectively.

It is known, [2], that the fibers of an almost contact metric submersion are invariant submanifolds of dimension $2(m-m')$ and ϕ induces on them an almost complex structure that we shall continue denoting by ϕ. We shall denote by g and ∇ the restrictions of $\bar{g}$ and $\overline{\nabla}$ to the fibers. Furthermore

$$\phi(V(M)) = V(M) \qquad H(M) = \phi(H(M)) \oplus \xi .$$

2 - The O'Neill configuration tensors

The O'Neill configuration tensors of the Riemannian submersion $f: M \to M'$ are given by

$$T_E F = \mathscr{H} \overline{\nabla}_{\mathscr{V}E} \mathscr{V} F + \mathscr{V} \overline{\nabla}_{\mathscr{V}E} \mathscr{H} F \qquad A_E F = \mathscr{V} \overline{\nabla}_{\mathscr{H}E} \mathscr{H} F + \mathscr{H} \overline{\nabla}_{\mathscr{H}E} \mathscr{V} F$$

where E, F are arbitrary vector fields on M.

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(**) Ricevuto: 10-IV-1985.

For an almost contact metric submersion $f: M \to M'$, it shall be useful to consider the tensor field B of type (1, 2) given by

$$B(E,F) = \mathscr{V}\overline{\nabla}_{\mathscr{H}E}(\phi\mathscr{H}F) - \mathscr{V}\overline{\nabla}_{\phi\mathscr{H}E}\mathscr{H}F + \mathscr{H}\overline{\nabla}_{\mathscr{H}E}(\phi\mathscr{V}F) - \mathscr{H}\overline{\nabla}_{\phi\mathscr{H}E}\mathscr{V}F$$

for E, F arbitrary vector fields on M.

This tensor B verifies the following properties, which can be easily deduced from the definition.

Proposition 2.1. *If X, Y and Z are horizontal vector fields on M, then:*

(i) $B(X,Y) = A_X\phi Y - A_{\phi X}Y$,

(ii) $B(X,Y) = B(Y,X)$,

(iii) $B(X,X) = 0$ *if and only if* $B(Y,Z) = 0$

(iv) $B(X,X) = B(\phi X, \phi X) - 2\eta(X)A_{\phi X}\xi$.

We now examine some properties of tensors T, A and B for an almost contact metric submersion, when in the total space different almost contact structures are considered [1], [5].

Theorem 2.1. *Let $f: M \to M'$ be an almost contact metric submersion and let M be a quasi-K-cosymplectic manifold (Q-K-C-manifold). Then:*

(a) $T_V\phi W = T_{\phi V}W$, (b) $T_{\phi V}X = -\phi T_V X$

(c) $A_X\phi Y = -A_Y\phi X = A_{\phi X}Y$,

(d) $A_X\phi V = -\phi A_X V = A_{\phi X}V$ (e) $B = 0$,

for all V, $W \in V(M)$ *and* X, $Y \in H(M)$.

Proof. Since the fibers of the submersion are quasi-Kählerian manifolds and $\eta(V) = 0$, for $V \in V(M)$, assertion (a) follows easily.

Assertion (b) is a consequence of (a) and the properties of tensor T.

(c) Since M is a Q-K-C-manifold

$$\overline{\nabla}_X(\phi)Y + \overline{\nabla}_{\phi X}(\phi)\phi Y = \eta(Y)\,\overline{\nabla}_{\phi X}\xi,$$

considering the horizontal part of this equation for case $X=Y$, we find

$$A_X\phi X-\phi A_X X-A_{\xi X}X-\phi A_{\xi X}\phi X=0$$

and due to the skew-symmetry of A on horizontal vectors, it results $A_X\phi X=0$ and the results follows now by polarization.

(d) By direct calculation we have $A_{\xi X}V=-\phi A_X V$.

We may assume that X is a basic horizontal vector and take the horizontal projection of the defining equation of Q-K-C-manifold

$$A_X\phi V=-\phi A_X V+\eta(A_X\phi V)\xi,$$

but $\overline{\nabla}_{\phi V}\xi=-\overline{\nabla}_V(\phi)\xi=-\phi\overline{\nabla}_V\xi$, which implies

$$\eta(A_\xi\phi V)=\eta(\mathscr{H}(\overline{\nabla}_{\phi V}\xi))=\eta(\overline{\nabla}_{\phi V}\xi)=0$$

and the result follows bearing in mind that $H(M)=\phi(H(M))\oplus\xi$ and $A_{\phi Y}\phi V=-\phi A_Y\phi V$.

Assertion (e) is a consequence of (c) and the definition of tensor B.

Theorem 2.2. *Let $f:M\to M'$ be an almost contact metric submersion and let M be a nearly-K-cosymplectic manifold (N-K-C-manifold). Then:*

(a) $T_V\phi W=\phi T_V W=T_{\phi V}W$ (b) $T_V\phi X=\phi T_V X=-T_{\phi V}X$ (c) $A=0$

for all $V,W\in V(M)$ and $X\in H(M)$.

Proof. (a) Since a N-K-C-manifold is a Q-K-C-manifold, it is sufficient to prove that $T_V\phi W=\phi T_V W$. But

$$T_V\phi V=\mathscr{H}\,\overline{\nabla}_V\phi V=\overline{\nabla}_V\phi V-\phi\overline{\nabla}_V V=\phi T_V V+\overline{\nabla}_V(\phi)V-\nabla_V(\phi)V$$

and since M is a N-K-C-manifold and the fibers of f are nearly Kähler manifolds, we find $T_V\phi V=\phi T_V V$ obtaining the result by polarization.

Assertion (b) follows from (a) of this theorem and (b) of Theorem 2.1.

(c) Consider the defining equation of N-K-C-manifold for X basic horizontal vector field

$$\overline{\nabla}_X\phi V-\phi\overline{\nabla}_X V+\overline{\nabla}_V\phi X-\phi\overline{\nabla}_V X=0$$

and take the horizontal projection of this equation

$$\overline{\nabla}_X \phi V - \phi \overline{\nabla}_X V + \overline{\nabla}_V \phi X - \phi \overline{\nabla}_V X = 0.$$

Using now assertion (d) of Theorem 2.1 and $V(M) = \phi(V(M))$, we find $A_X V = 0$, which is equivalent to $A = 0$.

The fact of that the vanishing of tensor A characterizes the integrability of the horizontal distribution, yields the

Corollary 2.1. *If $f: M \to M'$ is an almost contact metric submersion and M is a N-K-C-manifold, then the horizontal distribution is integrable.*

Corollary 2.2. *Let $f: M \to M'$ be an almost contact metric submersion and let M be a N-K-C-manifold. Then $T_V \phi E = \phi T_V E$, for $V \in V(M)$ and E an arbitrary vector field on M.*

Theorem 2.3. *Let $f: M \to M'$ be an almost contact metric submersion and let M be a normal manifold. Then:*

(a) $\phi\{T_V E - T_{\phi V} \phi E\} = T_V \phi E + T_{\phi V} E$ (b) $\phi\{A_X E - A_{\phi X} \phi E\} = A_X \phi E + A_{\phi X} E$

for $X \in H(M)$, $V \in V(M)$ and E a general vector field on M.

Proof. In order to prove (a) note that

$$T_V \phi E = \mathcal{H}\, \overline{\nabla}_V \phi(\mathcal{V} E) + \mathcal{V}\, \overline{\nabla}_V \phi(\mathcal{H} E) = \phi T_V E + \mathcal{H}\, (\overline{\nabla}_V(\phi)\, \mathcal{V} E) + \mathcal{V}(\overline{\nabla}_v(\phi)\, \mathcal{H} E)$$

and since M is a normal manifold [5], this implies that

$$T_V \phi E = \phi T_V E - \mathcal{H}\, (\overline{\nabla}_{\phi V} \mathcal{V} E) - \phi(\mathcal{H}\, \overline{\nabla}_{\phi V} \phi(\mathcal{V} E)) - \mathcal{V}(\overline{\nabla}_{\phi V} \mathcal{H} E) - \phi(\mathcal{V}(\overline{\nabla}_{\phi V}(\phi) \mathcal{H} E))$$

$$= \phi T_V E - \phi T_{\phi V} \phi E - T_{\phi V} E.$$

The assertion (b) is proved in an analogous way.

3 - Transference of $K_{i\phi}$-curvatures

In this section we shall be interested in answering the following question: if $f: M \to M'$ is an almost contact metric submersion with M satisfying the $K_{i\phi}$-

curvature identity, under what conditions M' satisfies the $K_{i\phi'}$-curvature identity?

Recall that an almost contact metric manifold satisfies the $K_{i\phi}$-curvature identity ($i = 1, 2, 3$), if for all $X, Y, Z, \quad W \in \mathscr{X}(M)$

$$K_{1\phi}: R(X, Y, Z, W) = R(\phi X, \phi Y, Z, W),$$

$$K_{2\phi}: R(X, Y, Z, W) = R(\phi X, \phi Y, Z, W) + R(\phi X, Y, \phi Z, W) + R(\phi X, Y, Z, \phi W),$$

$$K_{3\phi}: R(X, Y, Z, W) = R(\phi X, \phi Y, \phi Z, \phi W),$$

and it is verifies that

$$K_{1\phi}\text{-curvature} \Rightarrow K_{2\phi}\text{-curvature} \Rightarrow K_{3\phi}\text{-curvature}.$$

Theorem 3.1. *Let $f: M \to M'$ an almost contact metric submersion with M satisfying the $K_{3\phi}$-curvature identity.*

(i) *If $B = 0$ on horizontal vector fields, then M' satisfies the $K_{3\phi'}$-curvature.*

(ii) *If M' satisfies the $K_{3\phi'}$-curvature identity, then for horizontal vector fields X and Y on M we have $\|A_X Y\| = \|A_{\phi X}\phi Y\|$.*

Proof. For any Riemannian submersion

$$R(X, Y, Z, T) = R'(X_*, Y_*, Z_*, T_*) - 2\bar{g}(A_X Y, A_Z T) + \bar{g}(A_Y Z, A_X T) + \bar{g}(A_Z X, A_Y T).$$

So, $$R(\phi X, \phi Y, \phi Z, \phi T) = R'(\phi' X_*, \phi' Y_*, \phi' Z_*, \phi' T_*) - 2\bar{g}(A_{\phi X}\phi Y, A_{\phi Z}\phi T)$$
$$+ \bar{g}(A_{\phi Y}\phi Z, A_{\phi X}\phi T) + \bar{g}(A_{\phi Z}\phi X, A_{\phi Y}\phi T).$$

Since M satisfies the $K_{3\phi}$-curvature identity, to see (i), it shall be sufficient to prove $A_X Y = -A_{\phi X}\phi Y$. Note that $B = 0$ on $H(M)$ is equivalent to $A_X \phi Y = A_{\phi X} Y$ and then

$$A_{\phi X}\phi Y = -A_X Y + \eta(X) A_\xi Y.$$

Now bearing in mind that A is skew-simmetric on $H(M)$ and $H(M) = \phi(H(M)) \oplus \xi$, to prove (i) is sufficient to prove that $A_{\phi(L)}\xi = 0$, for all $L \in H(M)$; but this is a consequence of assertion (iv) in Proposition 2.1 and the proof is finished.

Conversely, if M' satisfies the $K_{3\phi'}$-curvature identity

$$-2\bar{g}(A_X Y, A_Z T) + \bar{g}(A_Y Z, A_X T) + \bar{g}(A_Z X, A_Y T)$$

$$= -2\bar{g}(A_{\phi X}\phi Y, A_{\phi Z}\phi T) + \bar{g}(A_{\phi Y}\phi Z, A_{\phi X}\phi T) + \bar{g}(A_{\phi Z}\phi X, A_{\phi Y}\phi T)$$

and the result follows setting $X = Z$, $Y = T$ in this expression.

Corollary 3.1. *Let $f: M \to M'$ be an almost contact metric submersion with M a quasi-K-cosymplectic manifold satisfying the $K_{3\phi}$-curvature. Then M' is also a Q-K-C-manifold satisfying the $K_{3\phi'}$-curvature identity.*

Theorem 3.2. *Let $f: M \to M'$ be an almost contact metric submersion with M satisfying the $K_{2\phi}$-curvature identity. Then*

(i) *M' satisfies the $K_{2\phi'}$-curvature identity if for X and Y horizontal vector fields*

$$A_X\phi Y = -A_{\phi X} Y \qquad or \qquad A_X \phi Y = A_{\phi X} Y = \phi A_X Y.$$

(ii) *If M' satisfies the $K_{2\phi'}$-curvature identity and $B = 0$ on $H(M)$, then $\|A_X Y\| = \|A_X \phi Y\|$ for X and Y horizontal vector fields on M.*

Proof. (i) Since M satisfies the $K_{2\phi}$-curvature identity, we have

$$R'(X_*, Y_*, Z_*, T_*)$$

$$- R'(\phi' X_*, \phi' Y_*, Z_*, T_*) - R'(\phi' X_*, Y_*, \phi' Z_*, T_*) - R'(\phi' X_*, Y_*, Z_*, \phi' T_*)$$

$$= 2\bar{g}(A_X Y, A_Z T) - \bar{g}(A_Y Z, A_X T) - \bar{g}(A_Z X, A_Y T) + \bar{g}(A_{\phi Y} Z, A_{\phi X} T) - 2\bar{g}(A_{\phi X}\phi Y, A_Z T)$$

$$+ \bar{g}(A_Z \phi X, A_{\phi Y} T) + \bar{g}(A_Y \phi Z, A_{\phi X} T) - 2\bar{g}(A_{\phi X} Y, A_{\phi Z} T) + \bar{g}(A_{\phi Z}\phi X, A_Y T)$$

$$+ \bar{g}(A_Y Z, A_{\phi X}\phi T) - 2\bar{g}(A_{\phi X} Y, A_Z \phi T) + \bar{g}(A_Z \phi X, A_Y \phi T),$$

but if A possesses one of the properties pointed above, $A_\phi \phi = 0$ and the right-hand side of this expression vanishes and we may conclude that M' satisfies the $K_{2\phi'}$-curvature identity.

(ii) Since M and M' satisfy the $K_{2\phi}$-curvature and $K_{2\phi'}$-curvature identity respectively, setting $Z = X$, $T = Y$ in the last expression and taking into account that $A_X Y = -A_Y X$ and $B(X, Y) = 0$, we have

$$A_{\phi X}\phi Y = -A_X Y \qquad A_{\phi X} Y = A_X \phi Y$$

and the result follows.

Corollary 3.2. *Let $f: M \to M'$ be an almost contact metric submersion with M a normal manifold satisfying the $K_{2\phi}$-curvature identity. Then, if B vanishes on the horizontal vector fields, M' is a normal manifold satisfying the $K_{2\phi'}$-curvature identity.*

Theorem 3.3. *Let $f: M \to M'$ be an almost contact metric submersion with M satisfying the $K_{1\phi}$-curvature identity. Then $A = 0$ if an only if M' satisfies the $K_{1\phi'}$-curvature identity and B vanishes on the horizontal vector fields on M.*

Proof. The necessity is obvious. Conversely, if M' satisfies the $K_{1\phi'}$-curvature identity,we obtain

$$3\|A_X Y\|^2 - 2\bar{g}(A_X Y, A_{\phi X}\phi Y) + \bar{g}(A_Y \phi X, A_X \phi Y) + \bar{g}(A_{\phi X} X, A_Y \phi Y) = 0$$

and, since $B(X, Y) = 0$, we deduce that

$$A_{\phi X} X = 0 \qquad A_{\phi X}\phi Y = -A_X Y \qquad -A_Y \phi X = A_X \phi Y$$

and so

$$5\|A_X Y\|^2 - \|A_X \phi Y\|^2 = 0.$$

Thus $A_X Y = 0$ after recalling the statement of Theorem 3.2.

Corollary 3.3. *Let $f: M \to M'$ be an almost contact metric submersion with M a quasi-K-cosymplectic manifold satisfying the $K_{1\phi}$-curvature identity. Then M' is a quasi-K-cosymplectic manifold satisfying the $K_{1\phi'}$-curvature identity if and only if $A = 0$.*

References

[1] D. E. Blair, *Contact manifolds in Riemannian Geometry*, Lecture Notes in Mathematics, **509** New York, 1976.

[2] D. Chinea, *Subvariedades invariantes y semi-invariantes de variedades casi-contacto y casi-complejas*, Publ. del Dpto. de Geometría y Topología. Univ. La Laguna (1983).

[3] A. GRAY and L. M. HERVELLA, *The sixsteen classes of almost Hermitian manifolds and their linear invariants*, Ann. Mat. Pura Appl. (IV) **123** (1980), 35-80.

[4] B. O'NEILL, *The fundamental equations of a submersion*, Michigan Math. J. **13** (1966), 459-469.

[5] J. A. OUBIÑA, *New classes of almost contact metric structures*, Publ. Math, Debrecen (to appear).

[6] B. WATSON, *Almost Hermitian submersions*, J. Differential Geom. **11** (1976), 147-165.

[7] B. WATSON and L. VANHECKE, *K_i-curvatures and almost Hermitian submersion*, Rend. Sem. Mat. Univ. Politec. Torino **36** (1977-78), 205-224.

Sommario

In questo lavoro si esaminano alcune proprietà dei tensori di configurazione di una sommersione metrica quasi contatto, quando nello spazio totale sono presenti varie strutture quasi contatto. Ciò consente di analizzare il comportamento delle K_{i}-curvature per effetto di una sommersione metrica quasi contatto.

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