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On the orthogonality of the Bessel polynomials (**)

Recently, Srivastava [5] has pointed out that the formula

$$(1)\qquad \int_0^\infty x^{1-a} e^{-x} y_m(x; a, 1)\, y_n(x; a, 1)\, dx$$

given by Hamza [3] is incorrect. The Bessel polynomial $y_n(x; a, b)$ is given by

$$(2)\qquad y_n(x; a, b) = 2F_0(-n, n+a-1; -; -x/b).$$

In this note, the scaling factor b has been put equal to unity without loss of generality. The Bessel polynomials were first studied on a systematic basis by Krall and Frink [4] who gave the orthogonality relation

$$(3)\qquad \int_{|x|=1} y_m(x; a, 1)\, y_n(x; a, 1)\, w(x)\, dx = \frac{(-1)^{n+1} n!\, \Gamma(a)}{(a+2n-1)\, \Gamma(a+n-1)} \delta_{m,n}$$

where $$w(x) = \sum_{m=0}^\infty \frac{\Gamma(a)}{\Gamma(a+m-1)} \left(-\frac{1}{z}\right)^m.$$

Now, the Bessel polynomials $y_n(x; a, 1)$ satisfy the differential equation

$$(4)\qquad x^2 y'' + (ax+1) y' - n(n+a-1)\, y = 0,$$

which may be written in the self-adjoint form

$$(5)\qquad [x^a \exp[-1/x]\, y']' - n(n+a-1)\, x^{a-2} \exp[-1/x]\, y = 0.$$

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In view of the interest shown in the Bessel polynomials, see Grosswald [2] for example, it seems worthwhile to investigate the matter of their orthogonality further.

From the general theory of Sturm-Liouville systems, the form of (5) leads us to consider integrals of the type

$$\int_C x^{a-2}\exp[-1/x]\,y_m(x;a,1)\,y_n(x;a,1)\,\mathrm{d}x \tag{6}$$

from the point of view of the orthogonality of these polynomials, always provided that the integral (6) converges. The contour C is to be so selected that

$$[x^a\exp[-1/x]]_C = 0\,. \tag{7}$$

Hence, this contour may of a simple loop beginning at the origin, encircling the point at $+\infty$ in the positive direction and returning to the origin. If $\mathrm{Re}(a) < 0$, C may also be taken to be a simple path joining 0 and $+\infty$.

Consider the integral

$$I_{m,n} = \int_0^{(\infty+)} x^{a-2}\exp[-1/x]\,y_m(x;a,1)\,y_n(x;a,1)\,\mathrm{d}x\,. \tag{8}$$

Replace each Bessel polynomial by its hypergeometric series representation (2) and integrate term by term. Hence

$$\begin{aligned} I_{m,n} &= \sum_{r,s=0}^{m,n}\frac{(-m)_r(-n)_s(m+a-1)_r(n+a-1)_s}{r!\,s!}(-1)^{r+s}\int_0^{(\infty+)} x^{a+r+s-2}\exp[-1/x]\,\mathrm{d}x \\ &= \frac{(-1)^a\,2\pi i}{\Gamma(a)}F_3(-m,-n,m+a-1,n+a-1;a;1,1) \end{aligned} \tag{9}$$

where F_3 is an Appell function of the third kind. See Exton [1] (page 24) for example.

The Appell function may be re-arranged in the form

$$F = \sum_{r=0}^{m}\frac{(-m)_r(m+a-1)_r}{(a)_r\,r!}\,2F_1(-n,n+a-1;a+r;1) \tag{10}$$

and on applying Vandermonde's theorem, the inner Gauss polynomial becomes

$$\frac{(r-n+1)_n}{(r+a)_n}. \tag{11}$$

If $r < n$, the numerator of this expression vanishes and since r runs from 0 to m, it follows that F vanishes when $m < n$. By symmetry, it will be seen that F also vanishes when $n < m$. If $m = n$, then all the terms of the series (10) vanish except that for which $r = n$ and we have

$$F = \frac{(-n)_n (n+a-1)_n}{(a)_n (n+a)_n} = \frac{(-1)^n n!(n+a-1)}{(a)_n (2n+a-1)}. \tag{12}$$

We may then write

$$I_{m,n} = \int_0^{(\infty+)} x^{a-2} \exp[-1/x]\, y_m(x; a, 1)\, y_n(x; a, 1)\, dx \tag{13}$$

$$= \frac{(-1)^{a+n} n!(n+a-1)(2\pi i)\,\delta_{m,n}}{\Gamma(a+n)(2n+a-1)},$$

which is a new orthogonality relation involving the Bessel polynomials, with an elementary weight function.

Similarly, if, for convergence, $\mathrm{Re}(a) < 1 - m - n$, we have

$$J_{m,n} = \int_0^{\infty} x^{a-2} \exp[-1/x]\, y_m(x; a, 1)\, y_n(x; a, 1)\, dx \tag{14}$$

$$= \frac{(-1)^n n!(n+a-1)\pi}{\Gamma(a+n)(2n+a-1)\,\mathrm{Sin}(\pi a)}\,\delta_{m,n}.$$

References

[1] H. Exton, *Multiple hypergeometric functions and applications*, Ellis Horwood Ltd. Chicester, U.K., 1976.

[2] E. Grosswald, *Bessel polynomials*, Lecture Notes in Mathematics **698** Springer, Berlin, Heidelberg, New York, 1978.

[3] A. M. Hamza, *Integrals involving Bessel polynomials*, Riv. Mat. Univ. Parma (3) **1** (1972), 41-46.

[4] H. L. Krall and O. Frink, *A new class of orthogonal polynomials: the Bessel polynomials*, Trans. Amer. Math. Soc. **65** (1949), 100-115.

[5] H. M. Srivastava, *A note on the Bessel polynomials*, Riv. Mat. Univ. Parma (4) **9** (1983), 207-212.

Abstract

A new orthogonality relation for the Bessel polynomials is given with an elementary weight function.

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