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Propagation and interference of waves in oscillatory heat conduction in composite media (**)

1 - Formulation and solution

The linear equation of heat conduction is used to study the propagation and interference of dispersive thermal waves in composite media by an oscillatory thermal forcing effect which is realized by imposing a time-dependent boundary condition of the form $\exp[(i\omega-\lambda^2)t]$ with the attenuation coefficient λ. The effect of the attenuation factor $\exp[-\lambda^2 t]$ is to decay the amplitude of the waves exponentially with time, the decay time being λ^{-2}. The wave structure thus generated is analyzed in regard to a composite flat plate made up of two regions, each of finite thickness. Thus, let the composite media be $-h\leq z\leq h$, of which the region $-h\leq z\leq 0$ is of one medium with K_1, ρ_1, c_1, α_1 and T_1 for conductivity, density, specific heat, diffusivity and temperature; $0\leq z\leq h$ is of other medium with the corresponding quantities as K_2, ρ_2, c_2, α and T_2.

After reducing the equations for linear heat conduction [2] and the boundary conditions in dimensionless form, we shall solve the following boundary-value problem: Determine the temperature fields $T_{1,2}(z, t)$ such that

(1) $$\frac{\partial^2 T_{1,2}}{\partial z^2}-\frac{\partial T_{1,2}}{\partial t}=0$$

(2) $$\frac{\partial T_1}{\partial z}=\sigma\frac{\partial T_2}{\partial z} \qquad T_1=T_2 \qquad \text{at} \quad z=0$$

(3) $$T_1(-1, t)=\exp(i\omega-\lambda^2)t \qquad T_2(1,t)=0 \quad t>0$$

where $\sigma=K_2/K_1$.

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Using the Laplace transform method the solution to the problem (1)-(3) gives the temperature fields $T_{1,2}(z, t)$ as

$$T_1 = \frac{1}{2(1+\sigma)} \operatorname{Re} \exp(-\mu t) \sum_{n=0}^{\infty} [(\cos n\pi + \sigma)(u_1 + u_3) + (\cos n\pi - \sigma)(u_2 + u_4)] \qquad -1 \le z \le 0 \tag{4}$$

$$T_2 = \frac{1}{1+\sigma} \operatorname{Re} \exp(-\mu t) \sum_{n=0}^{\infty} (v_1 - v_2 + v_3 - v_4) \qquad 0 \le z \le 1 \tag{5}$$

where:

$$\begin{aligned} &\mu = \lambda^2 - i\omega \\ &u_1 = \exp(-i(2n+1+z)\sqrt{\mu}) \operatorname{erfc}\left(\frac{2n+1+z}{2\sqrt{t}} - i\sqrt{\mu t}\right) \\ &u_2 = \exp(-i(2n+1-z)\sqrt{\mu}) \operatorname{erfc}\left(\frac{2n+1-z}{2\sqrt{t}} - i\sqrt{\mu t}\right) \\ &v_1 = \exp(-i(4n+1+z)\sqrt{\mu}) \operatorname{erfc}\left(\frac{4n+1+z}{2\sqrt{t}} - i\sqrt{\mu t}\right) \\ &v_2 = \exp(-i(4n+1-z)\sqrt{\mu}) \operatorname{erfc}\left(\frac{4n+1-z}{2\sqrt{t}} - i\sqrt{\mu t}\right) \end{aligned} \tag{6}$$

and u_3, u_4 are like u_1, u_2, while v_3, v_4 are like v_1, v_2 except that $-i$ is replaced by i. For $\lambda = \omega = 0$ this solution coincides with those given in [2] for the case when $T_1(-1, t) = H(t)$, $t > 0$, where $H(t)$ is the Heaviside unit function.

The wave patterns generated by the temperature fields T_1 and T_2 are analyzed in 2 (a) and (b) respectively. The case $\sigma = 1$ reduces the problem to that of an infinite plate occupying the region $-1 \le z \le 1$; the analysis of waveforms for this case is carried out in 2 (c). The problem for the semi-infinite composite region $-1 \le z < \infty$ is solved in 2 (d) and its wave structure is discussed there. The technique of [4] is used to analyze the wave structure in all these cases.

2 - Structure of waves

In order to analyze the propagation of waves determined by the temperature fields T_1, T_2 as defined by (4)-(6), we shall use the approximation for the function erfc $(p \pm iq)$ as given in [1]. Then each term u_1, u_2, u_3, u_4 and v_1, v_2, v_3, v_4 contains factors of the form $F(z \pm iq)$ and $G(z, t)$. We shall therefore analyze the wave motions as represented by waveforms of the type $G(z, t)F(z \pm ct)$. Even though certain terms will be found to contain more than one waveform, only one of them is found to yield a real wave-front in each of the u_i and v_i profiles inside the region under consideration $(i = 1, 2, 3, 4)$.

(a) *Temperature field* T_1. In the u_1-profile the only real wave-front exists at

$$z = -(2n+1) + \omega t/a_1 \text{ where } \sqrt{\mu} = a_1 - ia_2, \quad a_{1,2} = [(\lambda^4 + \omega^2)^{\frac{1}{2}}/2 \pm \lambda^2]^{\frac{1}{2}}.$$

This is a dispersive diffuse progressive (transverse) cosinusoidal wave with attenuation factor $\exp(-\lambda^2 t)$, wavenumber a_1, velocity ω/a_1 and amplitude

$$\zeta_1(z, t) = |(\cos n\pi + \sigma)\, g(p_1)/2(1+\sigma)| \exp[-a_2(2n+1+z)]$$

where

$$g(p_1) = \operatorname{erfc} p_1 - \frac{4p_1 \exp(-p_1^2)}{\pi} \sum_{k=0}^{\infty} \frac{\exp(-k^2/4)}{k^2 + 4p_1} - \frac{\exp(-p_1^2)}{2\pi p_1}$$

$$p_1 = \frac{2n+1+z}{2\sqrt{t}} - a_2\sqrt{t}$$

and it propagates in the time intervals $2na_1/\omega \leq t \leq (2n+1)a_1/\omega)$ for each $n \geq 0$. For a given n, this wave attains antinodes at the points $z = -(2n+1) + (m\pi + \omega t)/a_1$ at any time t, where $m = 0, +1, +2, \ldots$; thus, the wave attains maxima or crests when $a_1(2n+1+z) - \omega t = 2m\pi$ and minima or troughs when $a_1(2n+1+z) - \omega t = (2m+1)\pi$, which yields wavelength $l = 2\pi/a_1$. For a given n, all the points on the u_1-profile at a given time, whose abscissae differ by an integral multiple of l, have the same phase. Since the wavelength varies only gradually, by a small fraction of itself from one wave to the next, we can define a local phase $\phi(z, t) = \omega t - a_1(2n+1+z)$ which gives $\partial\phi/\partial z = -a_1$, $\partial\phi/\partial t = \omega$.

Thus, using the argument of [3], we find that the group velocity U for this cosinusoidal wave is given by $U = 4a_1\sqrt{\lambda^4 + \omega^2}/\omega$. It should be noted that the heat energy of this cosinusoidal wave is dispersed at the group velocity U and not at the wave speed $s_1 = \omega/a_1$.

The u_2-profile has only one real wave-front which exists at $z = 2n + 1 - \omega t/a_1$. This is again a progressive (transverse) cosinusoidal wave with the same attenuation factor, wavenumber, wavelength, wave speed and group velocity as for the u_1-profile, but it has velocity $-\omega/a_1$ and its amplitude is given by

$$\zeta_2(z,t) = \left| \frac{(\cos n\pi - \sigma)}{2(1+\sigma)} g(p_1') \right| \exp[-a_2(2n+1-z)]$$

for a given n, where

$$p_1' = \frac{2n+1-z}{2\sqrt{t}} - a_2\sqrt{t}\,.$$

It attains crests when $a_1(2n+1-z) - \omega t = 2m\pi$ and troughs when $a_1(2n+1-z) - \omega t = (2m+1)\pi$. Although the two cosinusoidal waves, given by u_1- and u_2-profile, differ in antinodes and amplitudes and $\zeta_1(z,t) > \zeta_2(z,t)$ for $-1 \le z < 0$ with $\zeta_1(0,t) = \zeta_2(0,t)$ (in fact, more precisely $\zeta_1(-z^*,t) = \zeta_2(z^*,t)$ where $0 \le z^* \le 1$), they move in opposite directions. Thus their superposition exhibits the following interference pattern: The wave, described by u_1-profile, which emanates at $t=0$ from the surface $z=-1$, reaches the surface $z=0$ in the time interval a_1/ω and a portion of it is then reflected while the remaining portion is refracted through the surface $z=0$ in the time interval a_1/ω and a portion of it is then reflected while the remaining portion is refracted through the surface $z=0$ in the region $0 < z \le 1$. This reflected wave is represented by the u_2-profile, which travels in the opposite direction in the complementary time intervals $(2n+1)\,a_1/\omega \le t \le (2n+2)\,a_1/\omega$ for each $n \ge 0$. The amount of refraction depends on the value of σ and the structure of the refracted wave is described below for the temperature fields T_2. Since the amplitudes $\zeta_1(z,t)$ and $\zeta_2(z,t)$ for the cosinusoidal waves and their reflections are different, the disturbance in the region $-1 \le z \le 0$ does not have the character of standing waves of any kind.

The u_3-profile shows only one real wave-front which exists at $z = -(2n+1) + \lambda^2 t/a_2$. It represents a dispersive diffusive progressive (transverse) nonsinusoidal wave (henceforth called a λ-wave) with wavenumber a_2, velocity λ^2/a_2, and amplitude

$$\zeta_3(z,t) = \left| \frac{(\cos n\pi + \sigma)}{2(1+\sigma)} g(p_2) \cos(\omega t + a_1(2n+1+z)) \right|$$

where

$$p_2 = \frac{2n+1+z}{2\sqrt{t}} + a_2\sqrt{t}.$$

The group velocity of this wave is $V = 4a_2\sqrt{\lambda^4+\omega^2}/\omega$. This wave occurs in the time intervals $2na_2/\lambda^2 \le t \le (2n+1)\,a_2/\lambda^2$. It may be noted that this wave decays exponentially and exists because of the presence of the attenuation factor λ in the boundary condition at $z = -1$; it vanishes for $\lambda = 0$.

The u_4-profile possesses only one real wave-front which exists at $z = 2n + 1 - \lambda^2 t/a_2$. It again represents a λ-wave with the same wavenumber and wave speed $s_2 = \lambda^2/a_2$ as that given by the u_3-profile, but it moves in the opposite direction with velocity $-\lambda^2/a_2$ and has amplitude

$$\zeta_4(z, t) = \left| \frac{(\cos n\pi - \sigma)}{2(1 + \sigma)} g(p_2') \cos(\omega t + a_1(2n + 1 - z)) \right|$$

where

$$p_2' = \frac{2n + 1 - z}{2\sqrt{t}} + a_2 \sqrt{t}.$$

It may be noted that $\zeta_3(z, t) > \zeta_4(z, t)$ for $0 < z \leq 1$, with $\zeta_3(0, t) = \zeta_4(0, t)$ (more precisely, $\zeta_3(-z^*, t) = \zeta_4(z^*, t)$, where $0 \leqslant z^* \leq 1$). Both ζ_3 and ζ_4 possess oscillatory character.

The u_3 and u_4 profiles on superposition show the following interference pattern: The λ-wave given by the u_3-profile, which starts at the surface $z = -1$ at the time $t = 0$ reaches the surface $z = 0$ in the time interval a_2/λ^2 and a portion of it (depending on the value of σ) is then reflected, while the remaining portion is refracted through the surface $z = 0$ in the region $0 < z \leq 1$. This reflected wave is represented by the u_4-profile; it has the same characteristics as those given by the u_3-profile, except that it propagates and disperses in the opposite direction in the complementary time intervals for $(n \geq 0)$ $(2n + 1) a_2/\lambda^2 \leq t \leq (2n + 2) a_2/\lambda^2$.

Since the amplitudes ζ_3 and ζ_4 are different in $-1 \leq z \leq 0$, the disturbance in this region does not have the character of standing waves of any kind. The refracted waves are discussed below. For given values of λ and ω, Table I represents the behavior of s_1, s_2, U, V, l and U/s_1.

(b) *Temperature field* T_2. The structure of waves represented by T_2 contains portions of refracted waves from T_1 through the separation surface $z = 0$ and their reflections at the surface $z = 1$. Since the heat energy is conducted faster from regions of lower conductivity to those of higher conductivity, the amount of refraction of thermal waves through the surface $z = 0$ depends on the value of σ. If $\sigma > 1$, then refraction is larger than that for $\sigma < 1$; in fact, if $\sigma \ll 1$, then refraction is negligibly small and the direct refracted waves through $z = 0$ and their subsequent reflections at $z = 1$ attenuate exceedingly rapidly.

The v_1-profile has only one real wave-front in the region $0 \leq z \leq 1$ and it exists at $z = -(4n + 1) + \omega t/a_1$. This is a dispersive diffusive progressive (transverse) cosinusoidal wave, refracted through the surface $z=0$ from the direct cosinusoidal wave, given by the u_1-profile, which started at the surface $z=-1$ at time $t=0$. It has the same attenuation factor, wavenumber, velocity, group velocity, wavelength as the direct cosinusoidal wave given by the u_1-profile, but its amplitude is

$$\zeta_5(z, t) = \left| \frac{g(b_1)}{1+\sigma} \right| \exp[-a_2(4n+1+z)]$$

where

$$b_1 = \frac{4n+1+z}{2\sqrt{t}} - a_2\sqrt{t};$$

it propagates in the time intervals $(4n+1)\,a_1/\omega \leq t \leq (4n+2)\,a_1/\omega$ for each $n \geq 0$.

The v_2-profile also has one real wave-front in the region $0 \leq z \leq 1$ and it exists at $z = 4n+3-\omega t/a_1$. This is again a progressive (transverse) cosinusoidal wave with the same attenuation factor, wavelength, wave speed and group velocity as given by the v_1-profile, but it has velocity $-\omega/a_1$ and its amplitude is

$$\zeta_6(z, t) = \left| \frac{g(b_1')}{1+\sigma} \right| \exp[-a_2(4n+3-z)]$$

where

$$b_1' = \frac{4n+3-z}{2\sqrt{t}} - a_2\sqrt{t}.$$

It attains its crests when $a_1(4n+3-z)-\omega t = 2m\pi$ and troughs when $a_1(4n+3-z)-\omega t = (2m+1)\pi$. Although the two cosinusoidal waves, given by v_1- and v_2-profile, differ in antinodes and amplitudes and $\zeta_5(z, t) > \zeta_6(z, t)$ for $0 < z \leq 1$, they move in opposite directions. Their superposition exhibits the following interference pattern: The wave described by the v_1-profile which was the refracted portion of the wave described by the u_1-profile at the surface $z=0$ and which started at $z=0$ at the time $t = a_1/\omega$ (counted from the initial time $t=0$ when the direct wave described by the u_1-profile emanated at the surface $z=-1$) reaches the surface $z=1$ in the additional time interval a_1/ω and is then reflected at the surface $z=1$. This reflected wave is represented by the v_2-profile, which travels in the opposite direction in the complementary time intervals $(4n+2)\,a_1/\omega \leq t \leq (4n+3)\,a_1/\omega$ for each $n \geq 0$ and reaches the surface $z=0$ where a portion of it is reflected and the remaining portion refracted, depending on the value of σ. Since the amplitudes $\zeta_1(z, t)$, $\zeta_2(z, t)$, $\zeta_5(z, t)$, and $\zeta_6(z, t)$ are different, the interference of these waves and their reflections and refractions at the surface $z = 0$ do not give rise to any kind of standing waves in the regions $0 \leq z \leq 1$ and $-1 \leq z \leq 0$.

The v_3-profile shows only one real wave-front which exists at $z = -(4n+1) + \lambda^2 t/a_2$. This is a λ-wave which is refracted through the surface $z = 0$ from the direct λ-wave given by the u_3-profile. This wave has the same wavenumber, velocity, group velocity as the direct λ-wave given by the u_3-profile, but its amplitude is

$$\zeta_7(z, t) = \left| \frac{g(b_2)}{1+\sigma} \cos(\omega t + a_1(4n+1+z)) \right|$$

where

$$b_2 = \frac{4n+1+z}{2\sqrt{t}} + a_2\sqrt{t}$$

and it travels in the region $0 \leq z \leq 1$ in the time intervals $(4n+1)\, a_2/\lambda^2 \leq t \leq (4n+2)\, a_2/\lambda^2$ for each $n \geq 0$.

The v_4-profile also has one real wave-front which exists at $z = 4n + 3 - \lambda^2 t/a_2$. This is again a λ-wave with the same wavenumber, wave speed and group velocity as that given by the v_3-profile, but it has velocity $-\lambda^2 t/a_2$ and its amplitude is given by

$$\zeta_8(z, T) = \left| \frac{g(b_2')}{1+\sigma} \cos(\omega t + a_1(4n+3-z)) \right|$$

where

$$b_2' = \frac{4n+3-z}{2\sqrt{t}} + a_2\sqrt{t}.$$

Although the two λ-waves, given by the v_3- and v_4-profile, differ in amplitudes and $\zeta_7(z, t) > \zeta_8(z, t)$ for $0 < z \leq 1$, they move in opposite directions. Their superposition shows the following interference pattern: The λ-wave, described by the v_3-profile which was the refracted portion of the λ-wave described by the u_3-profile at the surface $z = 0$ and which started at $z = 0$ at time $t = a_2/\lambda^2$ (counted from the initial time $t = 0$ when the direct λ-wave described by the u_3-profile emanated at $z = -1$) reaches the surface $z = 1$ in the additional time intervals λ^2/a_2 and is then reflected at the surface $z = 1$. This reflected λ-wave is represented by the v_4-profile, which travels in the opposite direction in the complementary time intervals $(4n+2)\,a_2/\lambda^2 \leq t \leq (4n+3)\,a_2/\lambda^2$ for each $n \geq 0$ and reaches the surface $z=0$ where a portion of it is reflected and the remaining portion refracted, depending on the value of σ. Again, since the amplitudes ζ_3, ζ_4, ζ_7 and ζ_8 are different, the interference of these waves and their reflections and refractions at the surface $z=0$ does not yield standing waves of any kind in the region $0 \leq z \leq 1$ or $-1 \leq z \leq 0$.

(c) *The case* $\sigma = 1$. The case when $\sigma = 1$ reduces the problem to that of a single infinite plate occupying the region $-1 \leq z \leq 1$. In this case the solution of the temperature field T is

$$T = \frac{1}{2}\,\mathrm{Re}\exp(-\mu t)\sum_{n=0}^{\infty}(v_1 - v_2 + v_3 - v_4) \qquad -1 \leq z \leq 1.$$

The structure of waves generated in this case is as follows: The v_1-profile has only one real wave-front in the region $-1 \leq z \leq 1$ and it exists at $z = -(4n+1) + \omega t/a_1$. It represents a dispersive diffusive progressive (transverse) cosinusoidal wave similar to the one in the v_1-profile of 2 (b), except that this wave propagates in the time intervals $4na_1/\omega \leq t \leq (4n+2)\,a_1/\omega$ for each $n \geq 0$. For $n=0$ this represents a direct wave starting at the surface $z=-1$, which reaches the surface $z=1$ in time ω/a_1 and is then reflected there. This reflected wave is represented by the v_2-profile which exists at $z=(4n+3)-\omega t/a_1$. This is again a dispersive diffusive progressive (transverse) cosinusoidal wave with the same characteristics as those given by the v_1-profile of this section, except that its velocity is $-\omega/a_1$, its amplitude is $\zeta_6(z, t)$, and it travels in the opposite direction. It represents the above-mentioned reflected wave which propagates in the complementary time intervals (for each $n \geq 0$) $(4n+2)\,a_1/\omega \leq t \leq (4n+4)\,a_1/\omega$.

The v_3-profile gives only one real wave-front in the region $-1 \leq z \leq 1$, which exists at $z = -(4n+1) + \lambda^2 t/a_2$. It represents a dispersive diffusive progressive (transverse) λ-wave of 2(b), except that it travels in the time intervals $4na_2/\lambda^2 \leq t \leq (4n+2)\,a_2/\lambda^2$ for each $n \geq 0$. For $n=0$ it represents the direct λ-wave which starts at the surface $z=-1$ at $t=0$ and reaches the surface $z=1$ in the time interval λ^2/a_2 when it is reflected there. This reflected wave is represented by the v_4-profile which exists at $z = 4n+3-\lambda^2 t/a_2$. This is again a transverse λ-wave with the same characteristics as those given by the v_3-profile of this section, except that its velocity is $-\lambda^2/a_2$, its amplitude is $\zeta_8(z, t)$, and it travels in the opposite direction. It represents the above-mentioned reflected λ-wave which progresses in the complementary time intervals (for each $n \geq 0$) $(4n+2)\,a_2/\lambda^2 \leq t \leq (4n+4)\,a_2/\lambda^2$. The λ-waves, represented by the v_3- and v_4-profile, vanish for $\lambda = 0$.

(d) *Semi-infinite composite region.* The solution for the semi-infinite composite region $-1 \leq z < \infty$, with separation surface at $z = 0$, such that $T_1(-1, t) = \exp[(i\omega - \lambda^2)t]$, $T_2(\infty, t) = 0, t > 0$, is given by

$$T_1 = \frac{1}{2} \text{ Re } \exp(-\mu t) \sum_{n=0}^{\infty} \beta^n (u_1 + u_3 - \beta(u_2 + u_4)) \qquad -1 \leq z \leq 0$$

$$T_2 = \frac{1}{2} \text{ Re } \exp(-\mu t) \sum_{n=0}^{\infty} \beta^n (u_1 + u_3) \qquad z > 0$$

where u_1, u_2, u_3, u_4 are defined by (6) and $\beta = (\sigma - 1)/(\sigma + 1)$.

The wave structure for this composite media is analogous to that described in 2 (a).

Table I

λ	ω	s_1	s_2	U	l	U/S_1	V
10.0	0.0	0.0	∞	∞	0.6283	∞	∞
	2.0	0.19	1000.06	2000.50	0.6282	10003.0	20.002
	4.0	0.39	500.09	1000.99	0.6281	2502.99	20.012
	10.0	0.99	200.25	402.49	0.6275	402.99	20.074
5.0	0.0	0.0	∞	∞	1.256	∞	∞
	2.0	0.39	125.09	250.99	1.255	627.99	10.02
	4.0	0.79	62.69	126.99	1.252	159.24	10.09
	10.0	1.96	25.47	54.87	1.233	27.96	10.56
1.0	0.0	0.0	∞	∞	6.28	∞	∞
	2.0	1.57	1.27	5.69	4.94	3.62	3.52
	4.0	2.49	0.80	6.59	3.92	2.64	5.15
	10.0	4.25	0.47	9.45	2.67	2.22	8.55
0.5	0.0	0.0	∞	∞	12.57	∞	∞
	2.0	1.87	0.26	4.29	5.90	3.61	3.51
	4.0	2.74	0.18	5.84	4.31	2.13	5.49
	10.0	4.42	0.11	9.06	2.77	2.05	8.84
0.1	0.0	0.0	∞	∞	62.83	∞	∞
	2.0	1.99	0.010	4.01	6.26	2.010	3.99
	4.0	2.82	0.007	5.66	4.43	2.005	5.64
	10.0	4.47	0.004	8.95	2.81	2.002	8.93

Conclusion

In view of the linear theory, the above analysis is valid for small n only.

It should be noted that although the two types of waves, one cosinusoidal and the other nonsinusoidal, are disperse waves, they are not at all resulting from the presence of any hyperbolic equations; they propagate because of the oscillatory forcing effect in the boundary conditions.

The case of the composite region $-h \leq z \leq a$, $a \neq h$, deserves a separate discussion.

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Summary

Exact solutions are obtained for temperature fields in composite media of two regions of finite as well as semi-infinite width when one surface is subjected to an oscillatory thermal forcing effect exp $[(i\omega - \lambda^2)t]$, $t > 0$, *while the other is kept at zero temperature. Two kinds of transverse dispersive waves, one cosinusoidal and the other nonsinusoidal, are generated; their reflections, refractions and interference are studied.*

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