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Orlicz metrics for weak convergence of distribution functions (**)

There has recently been a renewed interest in the field of metrics for the weak convergence of distribution functions on $\overline{R}$ [3], $[4]_{1,2}$, [6] and $\overline{R}^d$ $[4]_3$. A new family of such metrics is introduced in the present note which is based on the use of Orlicz spaces [1], [2]. This new family includes, as a special case, the family of metrics $d_{h,p}$, with $p>1$, introduced by M. D. Taylor [6] for the same purpose.

Let $p: R_+ \to R_+$ be a non-decreasing, right-continuous, integrable (on R_+) function such that $p(0)=0$, $p(t)>0$ if $t>0$ and $\lim_{x\to\infty} p(x) = +\infty$. The function $\varphi: R \to R_+$ defined by

$$\varphi(x) := \int_0^{|x|} p(t)\,dt$$

is said to be *a continuous Young function*. Notice that $\varphi(x)>0$ if $x>0$. The class of continuous Young functions will be denoted by $\mathscr{Y}$; its subset formed by the moderated continuous Young functions is denoted by $\mathscr{Y}_m$. A Young function is said to be *moderated* if there exists a real number $x_0>0$ such that $\varphi(2x) \leqq k\varphi(x)$ whenever $x \geqq x_0$ (in [1], φ is then said to verify the Δ_2-condition). The definition of Young function just given is slightly different from that of reference [1]. Those authors neglect to make explicit the assumption that p be integrable on R_+ (and hence, a fortiori, on $[0,x]$ for every $x \in R_+$) although they tacitly use it throughout their book. Without it, the class of Young functions would include functions defined as above and continuous except for at most one point after which the function must be identically equal to $+\infty$. In this note, we are interested only in continuous Young functions. Given a finite measure space $(\Omega, \mathscr{F}, \mu)$ and a continuous Young function φ we shall consider the Orlicz space

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$L^{\varphi}(\Omega, \mathcal{F}, \mu)$ (see [1] for the definitions). This is a Banach space under either the Orlicz or the Luxemburg norm. Since these two norms are equivalent [1], we shall write $\| \ \|_{\varphi}$ meaning either of them.

A sequence $\{f_n\} \subset L^{\varphi}$ is said to be *mean convergent* to f if $\lim_n \rho(f_n - f, \varphi) = 0$ where $\rho(g, \varphi) := \int \varphi \circ g \, d\mu$. Convergence in norm implies that in mean; the converse is true if φ is moderated.

In the sequel the following result will be needed. Despite its elementary nature, it is given here because I have been unable to trace an explicit formulation of it in the literature.

Lemma. *Convergence in mean implies convergence in measure.*

Proof. One has for every $\varepsilon > 0$, since $\varphi(\varepsilon) > 0$,

$$\int \varphi(f_n - f) \, d\mu \geqq \int_{\{|f_n - f| \geqq \varepsilon\}} \varphi(f_n - f) \, d\mu \geqq \varphi(\varepsilon) \mu\{|f_n - f| \geqq \varepsilon\}.$$

Since L^p spaces are, for $p > 1$, particular cases of Orlicz spaces, it is easy to generalize Taylor's version(s) of a distance on the spaces Δ of distribution functions [6]. If $\mathcal{H}$ denotes the class of homeomorphisms between the unit interval $I = [0, 1]$ and the extended real line $\overline{R}$ and if φ is a continuous Young function, it is possible to define a metric $d_{h,\varphi} : \Delta \times \Delta \to R_+$ via

$$d_{h,\varphi}(F, G) := \|F_0 h - G_0 h\|_{\varphi} \qquad (F, G \in \Delta) \tag{1}$$

where, from now on, $\| \ \|_{\varphi}$ denotes the norm on the Orlicz space $L^{\varphi} = L^{\varphi}(I, \mathcal{B}(I), \lambda)$, λ being the Lebesgue measure on (I, $\mathcal{B}$ (I)). Expression (1) defines on Δ a new family of metrics, one for each homeophormism h and for each continuous Young function. This new class includes Taylor's, in the sense that for $\varphi(x) = |x|^p/p$, with $p > 1$, either the Orlicz norm or the Luxemburg one is a multiple of the L^p-norm. Every metric defined by (1) metrizes, as will be seen shortly, the topology of weak convergenge on Δ, a property shared with Sibley's metric [5], [3], the metric d_k of $[4]_1$ and Taylor's metrics [6].

One is now ready to show the main result.

Theorem. *If $F_n, F \in \Delta$ $(n \in N)$, the following statements are equivalent:*

(a) $F_n \to F$ *weakly;*

(b) $\lim_n \rho(F_n \circ h - F \circ h, \varphi) = 0$ *for every homeomorphism* $h \in \mathcal{H}$ *and for every continuous Young function* $\varphi \in \mathcal{Y}$;

(c) $\lim_n d_{h,\varphi}(F_n, F) = 0$ *for every homeomorphism* $h \in \mathscr{H}$ *and for every moderated Young function* $\varphi \in \mathscr{Y}_m$;

(d) *there exist a homeomorphism* $h \in \mathscr{H}$ *and a continuous Young function* $\varphi \in \mathscr{Y}$ *such that* $\lim_n d_{h,\varphi}(F_n, F) = 0$.

Proof. (a) $\Rightarrow$ (b). By assumption $F_n(x) \to F(x)$ for every point x at which F is continuous, that is for every $x \in \overline{R}$ with the possible exception of a countable infinity of points. Since h is continuous and one-to-one, one has $(F_n \circ h)(t) \to (F \circ h)(t)$ for every $t \in I$ with the possible exception of the points in a countable subset of I. Thus in particular, $F_n \circ h \to F \circ h$ λ – a.e. As a consequence, $\varphi(F_n \circ h - F \circ h) \to 0$ λ – a.e. for every continuous Young function φ. Now $\varphi(F_n \circ h - F \circ h) \leqq \varphi(1) < \infty$ so that Lebesgue's theorem on dominated convergence yields the assertion.

(b) $\Rightarrow$ (c). The assumption states that $F_n \circ h$ tends to $F \circ h$ in mean; if φ is moderated, mean and norm convergence coincide ([1], Theorem 9.4) so that (c) obtains.

The implication (c) $\Rightarrow$ (d) is trivial.

(d) $\Rightarrow$ (a). By assumption $\|F_n \circ h - F \circ h\|_\varphi \to 0$ so that, by Lemma 9.2 in [1] and the lemma above, the sequence $\{F_n \circ h\}$ converges to $F \circ h$ in λ-measure, and since it is uniformly bounded, converges to it in $L^p = L^p(I, \mathscr{B}(I), \lambda)$ for every $p \in [1, \infty[$. Because of Theorem 1 in [6] this is equivalent to saying $F_n \to F$ weakly.

A different proof of the implication (d) $\Rightarrow$ (a) is possible, by making recourse to the concept of uniform integrability. The sequence $\{F_n \circ h\}$ belongs to L^1, converges to $F \circ h$ in λ-measure and is bounded in L^φ; but then it is uniformly integrable ([2], Theorem 1.4.4) and therefore , by Vitali's theorem (see, e.g., [2], Theorem 1.4.6), it converges to $F \circ h$ in L^1, which, as above, yields the assertion.

Notice also that the implication (d) $\Rightarrow$ (a) becomes trivial if statement (d) is replaced by the stronger one.

There exists a homeomorphism $h \in \mathscr{H}$ *such that* $d_{h,\varphi}(F_n, F) \to 0$ *either* (i) *for every* $\varphi \in \mathscr{Y}$ *or* (ii) *for every* $\varphi \in \mathscr{Y}_m$.

It suffices, in fact, to take $\varphi(x) = |x|^p/p$ $(p > 1)$ in order to obtain convergence in L^p and hence $F_n \to F$ through Taylor's theorem.

It should be noticed that the mapping $(F, G) \mapsto \rho(F \circ h - G \circ h, \varphi)$ does not define a metric on Δ; indeed, although it verifies the other axioms for a metric, it cannot satisfy the triangle inequality since no continuous Young functions φ is subadditive (i.e. $\varphi(x + y) \leqq \varphi(x) + \varphi(y)$ $\forall x, y \in R$) as would be required for that inequality to hold.

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Sommario

Le metriche degli spazi di Orlicz possono essere usate per metrizzare la topologia della convergenza debole per funzioni di ripartizione. Tali metriche includono come caso speciale la famiglia di distanze $d_{h,p}$ recentemente introdotte da Taylor allo stesso scopo $(p > 1)$.

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