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A contraction mapping principle and some applications (**)

1 – Luxemburg $[6]_1$ has proved a contraction mapping principle in a «generalized metric space» (not every two points have necessarily finite distance). Applications of the same idea can be found by many authors [1], [3], $[6]_2$, ..., [9]. The method used here is based on the concept of a generalized metric space with its distance function taking vector values. We present some variant of the Luxemburg theorem and use our result to establish the well-posedness of the Cauchy problem for the system $x' = F(t, x)$ with the righ-hand side in certain $\mathscr{L}^*$-spaces, which arise in a natural way (in particular, in spaces almost uniform convergence and pointwise convergence). For applications of original Luxemburg theorem to differential equations, see [2], $[6]_1$, [11], [14].

2 – Throughout this paper, $\mathbb{R}^k$ denote the k-dimensional Euclidean space, S the positive cone in $\mathbb{R}^k$, and S_∞ the set of k-tuples $(q_1, q_2, \ldots, q_k)$ with $0 \leqslant q_i \leqslant +\infty$ for $i = 1, 2, \ldots, k$. For $u = (u_1, u_2, \ldots, u_k)$ and $v = (v_1, v_2, \ldots, v_k)$ in S_∞, $u \leqq v$ is defined as usual i.e. $u_i \leqq v_i$ for each i. In S_∞ linear operations are defined as natural extensions of those $\mathbb{R}^k$.

We introduce the following definitions.

A *generalized metric space* (M, d) is a pair composed of a nonempty set M and a mapping d: $M \times M \to S_\infty$ satisfying for x, y, z in M the following conditions: (1) $d(x, y) = \theta$ if and only if $x = y$ (θ denote the zero of $\mathbb{R}^k$); (2) $d(x, y) = d(y, x)$; (3) $d(x, y) \leqq d(x, z) + d(z, y)$.

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Further, let us put

$$d^+(x,\ y) = \begin{cases} \|d(x,\ y)\| & \text{if } d(x,\ y) \in S \\ +\infty & \text{if } d(x,\ y) \in S_\infty \setminus S \end{cases}$$

for x, y in M. If every d^+-Cauchy sequence is d^+-convergent in M, then $(M,\ d)$ is called a *generalized complete metric space.*

Moreover, we shall use the notation of an $\mathscr{L}^*$-space [5]. The set $\mathscr{H}$ is called an *$\mathscr{L}^*$-space* if a certain class of sequences in $\mathscr{H}$ (the elements of this class are named *convergent sequences*) is distinguished in such a way that for every sequence (p_n) from this class there exists an element $p = \lim_{n\to\infty} p_n$ in $\mathscr{H}$ having the following properties: (1) if $\lim_{n\to\infty} p_n = p$ and $k_1 < k_2 < \ldots$, then $\lim_{n\to\infty} p_{k_n} = p$; (2) if $p_n = p$ for each n, then $\lim_{n\to\infty} p_n = p$; (3) if the sequence (p_n) is not convergent to p, then it contains a subsequence in which every subsequence fails to converge to p.

Let $\mathscr{U}$ and $\mathscr{V}$ be two $\mathscr{L}^*$-spaces. A mapping f of $\mathscr{U}$ into $\mathscr{V}$ is called *continuous* if for each point $p_0 \in \mathscr{U}$ and each sequence (p_n) in $\mathscr{U}$ converging to p_0 we have $\lim_{n\to\infty} f(p_n) = f(p_0)$.

3 – Suppose we are given: $(X,\ d)$ a generalized complete metric space; Y an $\mathscr{L}^*$-space; T a mapping from $X \times Y$ into X; z_0 an element in X such that $d(z_0,\ T(z_0,\ y)) \in S$ for each $y \in Y$.

Our results reads as follows.

Theorem. *Let $d(T(x_1,\ y),\ T(x_2,\ y)) \leqq L(d(x_1,\ x_2))$ for $x_1,\ x_2 \in X$ with $d(x_1,\ x_2) \in S$ and $y \in Y$, where L is a bounded linear operator on $\mathbb{R}^k$ whose spectral radius less than* 1 *and $L[S] \subset S$. Then, there exists a unique function φ: $Y \to X$ having for each $y \in Y$ the following properties:*

(i) *$T(\varphi(y),\ y) = \varphi(y)$ and $d(z_0,\ \varphi(y)) \in S$.*

(ii) *Every sequence of successive approximations $x_n = T(x_{n-1},\ y)$ $(n = 1,\ 2,\ \ldots)$ with $d(x_0,\ z_0) \in S$ is d^+-convergent to $\varphi(y)$.*

Moreover, if the function $T(\varphi(y),\ \cdot)$ maps continuously Y into $(X,\ d^+)$ and $d(\varphi(y_1),\ \varphi(y_2)) \in S$ for y_1, y_2 in Y, then φ is continuous from the $\mathscr{L}^*$-space Y into $(X,\ d^+)$.

Proof. Let X_0 denote the set of all x in X such that $d(x, z_0) \in S$. Then (X_0, d^+) is a complete metric space and $x \mapsto T(x, y)$ (with fixed y in Y) maps X_0 into itself. Applying Theorem 13.1.2. from [10], we can conclude the proof of the first part of our theorem.

Denote by $r(L)$ the spectral radius of operator L. Choose $\varepsilon > 0$ so that $r(L) + \varepsilon < 1$. Let us denote by $\|\cdot\|_\varepsilon$ a norm equivalent to $\|\cdot\|$ and such that $|L|_\varepsilon \leq \varepsilon + r(L)$ (see [10], Theorem 2.2.8; here $|L|_\varepsilon$ is the norm of L generated by $\|\cdot\|_\varepsilon$). Now, assume that (y_n) is a convergent sequence in Y with limit y_0. It is easy to prove that

$$\|d(\varphi(y_n), \varphi(y_0)) - L(d(\varphi(y_n), \varphi(y_0)))\|_\varepsilon \leq c\,\|d(T(\varphi(y_0), y_n), T(\varphi(y_0), y_0))\|_\varepsilon$$

$(n = 1, 2, \ldots)$ where c is constant. Hence

$$\begin{aligned} &\|d(\varphi(y_n), \varphi(y_0))\|_\varepsilon \\ &\leq \|d(\varphi(y_n), \varphi(y_0)) - L(d(\varphi(y_n), \varphi(y_0)))\|_\varepsilon + |L|_\varepsilon \|d(\varphi(y_n), \varphi(y_0))\|_\varepsilon \\ &\leq c\,\|d(T(\varphi(y_0), y_n), T(\varphi(y_0), y_0))\|_\varepsilon + (\varepsilon + r(L))\|d(\varphi(y_n), \varphi(y_0))\|_\varepsilon ; \end{aligned}$$

therefore $\|d(\varphi(y_n), \varphi(y_0))\|_\varepsilon \to 0$ as $n \to \infty$ and we have finished.

4 – Let $I = [0, a]$. Assume that L_{ij} $(i, j = 1, 2, \ldots, k)$ are functions on I with $0 \leq L_{ij}(t) \leq +\infty$, λ_i $(i = 1, 2, \ldots, k)$ are bounded real functions on I such that $\lambda_i(t) > 0$ for $0 < t \leq a$ and the functions $\lambda_j L_{ij}$ $(i, j, = 1, 2, \ldots, k)$ are integrable on I.

Let us denote: $C(I)$ the class of all continuous functions from I to R^k; $\mathscr{F}$ the set of continuous functions $F = (f_1, f_2, \ldots, f_k)$ from $I \times \mathrm{R}^k$ into R^k such that

$$|f_i(t, u) - f_i(t, v)| \leq \sum_{j=1}^{k} L_{ij}(t)\,|u_j - v_j| \qquad (i = 1, 2, \ldots, k)$$

for $0 < t \leq a$ and $u = (u_1, u_2, \ldots, u_k)$, $v = (v_1, v_2, \ldots, v_k)$ in R^k.

Moreover, let us put

$$a_{ij} = \sup_{0 < t \leq a} \frac{1}{\lambda_i(t)} \int_0^t \lambda_j(s)\, L_{ij}(s)\, \mathrm{d}s \qquad (i, j = 1, 2, \ldots, k)\,.$$

Further, let

$$T(x;\ F)=(T_1(x;\ F),\ T_2(x;\ F),\ \dots,\ T_k(x;\ F))$$

for $F=(f_1,\ f_2,\ \dots,\ f_k)\in\mathscr{F}$ and $x\in C(I)$, where

$$T_i(x;\ F)(t)=\int_0^t f_i(s,\ x(s))\,\mathrm{ds}\qquad (i=1,\ 2,\ \dots,\ k)$$

on I. For $x=(x_1,\ x_2,\ \dots,\ x_k)$ and $y=(y_1,\ y_2,\ \dots,\ y_k)$ in $C(I)$, we define

$$d(x,\ y)=\sup_{\substack{i=1,\dots,k\\ 0<t\leqslant a}}\frac{|x_i(t)-y_i(t)|}{\lambda_i(t)}.$$

Suppose that there exists a function $z=(z_1^0,\ z_2^0,\ \dots,\ z_k^0)\in C(I)$ such that for every fixed $(f_1,\ f_2,\ \dots,\ f_k)$ in $\mathscr{F}$ we have

$$z_i^0(t)-\int_0^t f_i(s,\ z(s))\,\mathrm{ds}=0(\lambda_i(t))\qquad (i=1,\ 2,\ \dots,\ k)$$

for $0<t\leqslant a$. Assume in addition that the set $\mathscr{F}$ is given some $\mathscr{L}^*$-space structure. Furthermore, let the following condition hold

(*) For every x in $C(I)$, with $d(x,\ z)\in S$, the transformation $F\mapsto T(x;\ F)$ from $\mathscr{F}$ into $(C(I),\ d^+)$ is continuous.

Example. Let the set $\mathscr{F}$ be $\mathscr{L}^*$-space endowed with the convergence $\lim_{n\to\infty}(f_1^{(n)},\ f_2^{(n)},\ \dots,\ f_k^{(n)})=(f_1^{(0)},\ f_2^{(0)},\ \dots,\ f_k^{(0)})$ means that

$$\lim_{n\to\infty}\sup_{0<t\leqslant a}\frac{1}{\lambda_i(t)}\int_0^t|f_i^{(n)}(s,\ x(s))-f_i^{(0)}(s,\ x(s))|\,\mathrm{ds}=0$$

$(i=1,\ 2,\ \dots,\ k)$ for every $x\in C(I)$. It is easy to check that if

$$\sup_{0<t\leqslant a}\frac{1}{\lambda_i(t)}\int_0^t\lambda_i(s)\,\mathrm{ds}<\infty\qquad (i=1,\ 2,\ \dots,\ k)$$

then the condition (+) is satisfied.

By (PC) we shall denote the problem of finding a solution of the differential equation $x' = F(t, x)$ (here $F \in \mathscr{F}$) satisfying the initial condition $x(0) = \theta$. This problem is equivalent to solving the equation $x = T(x; F)$ in the generalized complete metric space $(C(I), d)$. Applying our theorem we obtain the following

Proposition. *Suppose that $[a_{ij}]$ is the matrix with the spectral radius less than 1. Then, for an arbitrary $F \in \mathscr{F}$, there exists a unique function x_F satisfying problem* (PC) *on I, $d(x_F, z) \in S$ and x_F is given by the d^+-converging sequence*

$$y_n(t) = \int_0^t F(s, y_{n-1}(s))\, ds \qquad (n = 1, 2, \ldots)$$

with $y_0 \in C(I)$ such that $d(y_0, z) \in S$. Moreover, $F \mapsto x_F$ maps continuously $\mathscr{F}$ into $(C(I), d^+)$.

Proof. Let $F = (f_1, f_2, \ldots, f_k) \in \mathscr{F}$. First observe that $d(z, T(z; F)) \in S$. Further, for $1 \leqslant i \leqslant k$, $0 < t \leqslant a$ and $x = (x_1, x_2, \ldots, x_k)$, $y = (y_1, y_2, \ldots, y_k)$ in $C(I)$, we have

$$|T_i(x; F)(t) - T_i(y; F)(t)| \leqslant \int_0^t \sum_{j=1}^k L_{ij}(s)\, |x_j(s) - y_j(s)|\, ds \,;$$

hence, if $\sup\limits_{0<t\leqslant a} \frac{1}{\lambda_i(t)} |x_i(t) - y_i(t)| < \infty$ then

$$\sup_{0<t\leqslant a} \frac{1}{\lambda_i(t)} |T_i(x); F)(t) - T_i(y; F)(t)| \leqslant \sum_{j=1}^k a_{ij} \sup_{0<t\leqslant a} \frac{1}{\lambda_j(t)} |x_j(t) - y_j(t)| \,.$$

Let L denote the linear operator generated by matrix $[a_{ij}]$. From the least inequality, we get $d(T(x; F), T(y; F)) \leqq L(d(x, y))$ for $F \in \mathscr{F}$ and x, y in $C(I)$ with $d(x, y) \in S$. To complete the proof it is enough to apply Theorem.

Let us notice that the uniqueness conditions of Rosenblatt-Krasnoselskii-Krein type (see [4], [12]) used in $[6]_1$ and $[6]_2$ imply the assumptions of Proposition. See also [2], [11], [13] and [14].

5 – We are now going to give some corollaries to the above proposition.

Let $\mathscr{F}$ and T be as in 4. Denote by $\mathscr{H}$ the set of all F in $\mathscr{F}$ with $L_{ij}(t) \equiv A_{ij}$ on I. $\mathscr{H}$ will be considered as $\mathscr{L}^*$-space, $C(I)$ with the usual supremum metric, and assume that for every $x \in C(I)$ the transformation $F \mapsto T(x; F)$ maps continuously $\mathcal{H}$ into $C(I)$.

For example, $\mathcal{H}$ endowed with almost uniform convergence is an $\mathcal{L}^*$-space satisfying the condition (*). If the set $\mathcal{H}$ is uniformly bounded and endowed with pointwise convergence, then using the Lebesgue bounded convergence theorem we obtain that (*) hold.

Let us put $\lambda_i(t) = \exp(rt)$ $(i = 1, 2, \ldots, k)$ for $t \in I$, where $r > 0$ is a constant. Then

$$a_{ij} = A_{ij} \cdot \sup_{t \in I} \exp(-rt) \int_0^t \exp(rs)\,ds < r^{-1} A_{ij} \qquad (i, j = 1, 2, \ldots, k)$$

and therefore there exists a constant r such that $[a_{ij}]$ is a matrix with the spectral radius less than 1. By Proposition, we obtain the following result.

For an arbitrary $F \in \mathcal{H}$ there exists a unique function x_F satisfying problem (PC) *on I, and $F \mapsto x_F$ maps continuously $\mathcal{H}$ into $C(I)$.*

Now, let $\mathcal{U}$ denote the set of all $(f_1, f_2, \ldots, f_k) \in \mathcal{F}$ with $L_{ij}(t) = A_{ij}/t$ and such that

$$|f_i(t, u)| \leq M_i \cdot t^q \qquad (i = 1, 2, \ldots, k) \qquad \text{for } (t, u) \in I \times \mathbf{R}^k$$

where $q > -1$, and A_{ij}, $M_i (i, j = 1, 2, \ldots, k)$ are constants. By $\mathcal{V}$ we represent the set of all real continuous functions f on $I \times \mathbf{R}^k$ such that

$$|f(t, u) - f(t, v)| \leq \sum_{i=1}^{k} \frac{Q_i}{t^{k-i+1}} |u_i - v_i|$$

for $0 < t \leq a$ and $u = (u_1, u_2, \ldots, u_k)$, $v = (v_1, v_2, \ldots, v_k)$ in $\mathbf{R}^k$.

The set $\mathcal{U}$ will be considered with convergence as in Example where $\lambda_i(t) = t^{q+1}$ $(i = 1, 2, \ldots, k)$ on I. In $\mathcal{V}$ we introduce the following convergence $\lim_{n\to\infty} f^{(n)} = f^{(0)}$ means that

$$\lim_{n\to\infty} \sup \{\frac{1}{t} |f^{(n)}(t, u) - f^{(0)}(t, u)| : 0 < t \leq a,\ u \in \Omega\} = 0$$

for every compact Ω in $\mathbf{R}^k$.

Let

$$z^0(t)=\frac{t^{q+1}}{q+1}(M_1,\ M_2,\ \dots,\ M_k) \qquad w^0(t)=(t^k,\ t^{k-1},\ \dots,\ t)$$

for $t\in I$. Denote by ρ, σ the above distance function d with $\lambda_i(t)=t^{q+1}$ and $\lambda_i(t)=t^{k-i+1}$ $(i=1,2,\dots,k)$ on I respectively. We obtain the following corollaries:

If the matrix $(1+q)^{-1}[A_{ij}]$ *has the spectral radius less than* 1, *then for* $F\in\mathscr{U}$ *there exists a unique function* x_F *satisfying* (PC) *on* I *and* $\rho(z^0,\ x_F)\in S$. *The function* $F\mapsto x_F$ *maps continuously* $\mathscr{U}$ *into* $(C(I),\ \rho^+)$. *Suppose that*

$$\sum_{i=1}^{k}((k-i+1)!)^{-1}Q_i<1\ .$$

Then for $f\in\mathscr{V}$ *there exists a unique k-times differentiable on I function* y_f *such that*

$$y_f(0)=y_f'(0)=\dots=y_f^{(k-1)}(0)=0\ , \qquad y_f^{(k)}(t)=f(t,\ y_f(t),\ y_f'(t),\ \dots,\ y_f^{(k-1)}(t))$$

for $t\in I$ *and* $\sigma(w^0,\ z_f)\in S$, *where* $z_f(t)=(y_f(t),\ y_f'(t),\ \dots,\ y_f^{(k-1)}(t))$ *on* I.

Moreover, the function $f\mapsto z_f$ *maps continuously* $\mathscr{V}$ *into* $(C(I),\ \sigma^+)$.

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