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The method of the small parameter in the theory of differential equations with impulse effect (**)

1 - Introduction

Systems with impulses are an adequate mathematical model of a number of phenomena and processes of the concern of physics, chemistry, control theory, radiotechnology and so on. The first contributions on the mathematical theory of systems with impulses are the papers of V. D. Millman and A. D. Mishkis $[1]_{1,2}$. The theory of systems of differential equations with impulses advances comparatively slowly in spite of the considerable interest displayed to it. This is due to a number of difficulties implied by some specific features of these systems.

One of the traditional problems considered by the qualitative theory of differential equations is related to the existence of periodic solutions.

The present paper proposes the Poincaré method for finding the periodic solutions of differential-difference systems with impulses.

2 - Statement of the problem. General assumptions

Consider the system with impulse effect at fixed moments

$$\dot{x} = f(t,\ x(t),\ x(t-h)) \qquad t \neq t_i$$

(1)

$$(\Delta x)_{t=t_i} = I_i(x(t_i),\ x(t_i - h))$$

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where $x\in R$, $f\colon R\times R\times R\to R$, $I_i\colon R\times R\to R$, $i\in Z$, $t_i\in R$ are fixed points such that $t_{i+1}>t_i$, $\lim_{i\to\pm\infty} t_i=\pm\infty$, $(\Delta x)_{t=t_i}=x(t_i+0)-x(t_i-0)$, Z is the set of integers.

Def. 1. A solution of system (1) will be called *the piecewise continuous function* $\varphi\colon \mathscr{T}\to R$ *with first order discontinuity points as* $t_i\in\mathscr{T}$, which, for $t\in\mathscr{T}$, $t\neq t_i$, satisfies the equation

$$\dot{x}=f(t,\ x(t),\ x(t-h))$$

and for $t=t_i\in\mathscr{T}$ it fulfills the jump condition

$$x(t_i+0)-x(t_i-0)=I_i(x(t_i-0)),\ x(t_i-h-0)) \qquad \text{where } \mathscr{T}\subset R.$$

Def. 2. Under value of the solution $\varphi(t)$ at the point τ we will understand

$$\varphi(\tau)=\varphi(\tau-0)=\lim_{\varepsilon\downarrow 0}\varphi(\tau-\varepsilon)\ .$$

In view of Def. 1 and Def. 2 the solution $\varphi(t)$ of system (1) is a piecewise continuous function with first order discontinuity points at t_i, for which

$$\varphi(t_i)=\lim_{\varepsilon\downarrow 0}\varphi(t_i-\varepsilon)\ .$$

Consider system (1) for $h=0$

$$\begin{aligned}&\dot{x}=f(t,\ x(t),\ x(t)) \qquad t\neq t_i\\ &(\Delta x)_{t=t_i}=I_i(x(t_i),\ x(t_i))\ .\end{aligned} \tag{2}$$

The system (2) will be called *generating system of system* (1). Denote one of its solutions by $\psi(t)$. Define the *system in variations with respect to* $\psi(t)$, as follows

$$\begin{aligned}&\dot{y}\ =g(t)\,y \qquad t\neq t_i\\ &(\Delta x)_{t=t_i}=Q_i\,y(t_i)\end{aligned} \tag{3}$$

where

$$\begin{aligned}g(t)&=\Big(\frac{\partial f(t,\ x,\ x)}{\partial x}\Big)_{x=\psi(t)}\\ Q_i\ &=\Big(\frac{\partial I_i(x,\ x)}{\partial x}\Big)_{x=\psi(t_i),\ i\in Z}\ .\end{aligned} \tag{4}$$

Def. 3. The solution $\psi(t)$ of system (2) is called an *isolated T-periodic solution* if:

1. The function $\psi(t)$ is T-periodic.

2. The inequality $X(T)\neq 1$, holds where $X(t)$ is the fundamental solution of the system in variations with respect to $\psi(t)$ (3).

Let the following group of conditions (A) be fulfilled:

A1. The generating system (2) has an isolated T-periodic solution $\psi(t)$ and a bounded open set $D\subset R$ exists, such that $\psi(t)\in D$.

A2. The function f: $R^3\to R$ is defined, continuous, periodic by its first argument with period T, it is twice differentiable by its second and third argument in the domain $G=\{(t,\ x,\ y)\colon\ t\in R,\ x,\ y\in D\}$ and its second derivatives satisfy the Lipschitz condition.

A3. The functions I_i: $R^2\to R$ are twice continuously differentiable in the domain $D\times D$, $i\in Z$.

A4. A number $p>0$ exists such that $I_{i+p}(x,\ y)=I_i(x,\ y)$, $x,\ y\in D$ and $t_{i+p}=t_i+T$, $i\in Z$.

A5. For every $i\in Z$ the inequality $1+Q_i\neq 0$ holds.

3 - Main results

I. *Construction of subsequent approximations.* We will ask for a solution of system (1), periodic with period T, as a limit of a sequence of T-periodic functions.

In system (1) we substitute the unknown function by the formula

$$x(t)=y(t)+\psi(t)\ . \tag{5}$$

The following equations for the function $y(t)$ are obtained

$$\begin{aligned}\dot y&=f(t,\ \psi(t)+y(t),\ \psi(t)+y(t))-f(t,\ \psi(t),\ \psi(t))\\&\qquad+\Delta f(t,\ x(t),\ x(t-h))\qquad t\neq t_i\\(\Delta y)_{t=t_i}&=I_i(y(t_i)+\psi(t_i),\ y(t_i)+\psi(t_i))-I_i(\psi(t_i),\ \psi(t_i))+\Delta I_i(x(t_i),\ x(t_i-h))\end{aligned} \tag{6}$$

where

$$\begin{aligned}\Delta f(t,\ x(t),\ x(t-h))&=f(t,\ x(t),\ x(t-h))-f(t,\ x(t),\ x(t))\\\Delta I_i(x(t_i),\ x(t_i-h))&=I_i(x(t_i),\ x(t_i-h))-I_i(x(t_i),\ x(t_i))\ .\end{aligned} \tag{7}$$

Equations (6) can be written in the form

$$\begin{aligned} &\dot{y}=g(t)\,y+Y(t,\ y)+\Delta f(t,\ x(t),\ x(t-h)) \qquad t\neq t_i \\ (8) \qquad &(\Delta y)_{t=t_i}=Q_i\,y(t_i)+\mathscr{T}_i(y(t_i))+\Delta I_i(x(t_i),\ x(t_i-h)) \end{aligned}$$

where $g(t)$ and Q_i are defined by the equalities (4), while

$$\begin{aligned} &Y(t,\ y)=f(t,\ \psi(t)+y,\ \psi(t)+y)-f(t,\ \psi(t),\ \psi(t))-g(t)\,y \\ (9) \qquad &\mathscr{T}_i(y)=I_i(\psi(t_i)+y,\ \psi(t_i)+y)-I_i(\psi(t_i),\ \psi(t_i))-Q_i\,y\ . \end{aligned}$$

Besides, the equalities

$$\Delta f(t,\ x(t),\ x(t))=0 \qquad \Delta I_i(x(t_i),\ x(t_i))=0$$

$$\lim_{y\to 0}\frac{1}{y}Y(t,\ y)=0 \qquad Y(t,\ 0)=0 \qquad \mathscr{T}_i(0)=0$$

hold.

Put $y_0=0$ and define the approximation $y_1(t)$ as a T-periodic solution of the system

$$\dot{y}=g(t)\,y+Y(t,\ y_0)+\Delta f(t,\ \psi(t),\ \psi(t-h)) \qquad t\neq t_i$$

$$(\Delta y)_{t=t_i}=Q_i\,y(t_i)+\mathscr{T}_i(y_0)+\Delta I_i(\psi(t_i),\ \psi(t_i-h)) \qquad \text{or}$$

$$\begin{aligned} &\dot{y}=g(t)\,y+\Delta f_0(t,\ h) \qquad t\neq t_i \\ (10) \qquad &(\Delta y)_{t=t_i}=Q_i\,y(t_i)+\Delta I_i^0(t_i,\ h) \end{aligned}$$

where

$$\begin{aligned} &\Delta f_0(t,\ h)=\Delta f(t,\ \psi(t),\ \psi(t-h)) \\ (11) \qquad &\Delta I_i^0(t_i,\ h)=\Delta I_i(\psi(t_i),\ \psi(t_i-h))\ . \end{aligned}$$

In view of condition A1, the system (10) has a unique T-periodic solution

$$y_1(t) = \int_0^T G(t, \tau)\,\Delta f_0(\tau, h)\,d\tau + \sum_{0<t_i<T} G(t, t_i)(1+Q_i)^{-1}\,\Delta I_i^0(t_i, h)$$

where

$$G(t, \tau) = \begin{cases} X(t)(1-X(T))^{-1}X^{-1}(\tau) & 0 \leqq \tau \leqq t \leqq T \\ X(t+T)(1-X(T))^{-1}X^{-1}(\tau) & 0 \leqq t < \tau \leqq T\,. \end{cases} \tag{12}$$

We analogously determine the K-th approximation $y_K(t)$, $K \geqq 2$ as a T-periodic solution of the system

$$\begin{aligned} &\dot{y} = g(t)\,y + Y(t, y_{K-1}) + \Delta f_{K-1}(t, h) \qquad t \neq t_i \\ &(\Delta y)_{t=t_i} = Q_i\,y(t_i) + \mathscr{I}_i(y_{K-1}(t_i)) + \Delta I_i^{K-1}(t_i, h) \end{aligned} \tag{13}$$

where

$$\Delta f_{K-1}(t, h) = \Delta f(t, \psi(t) + y_{K-1}(t), \psi(t-h) + y_{K-1}(t-h))$$

$$\Delta I_i^{K-1}(t_i, h) = \Delta I_i(\psi(t_i) + y_{K-1}(t_i), \psi(t_i-h) + y_{K-1}(t_i-h))\,.$$

Then

$$\begin{aligned} y_{K-1}(t) = &\int_0^T G(t, \tau)[Y(\tau, y_{K-1}(\tau)) + \Delta f_{K-1}(\tau, h)]\,d\tau \\ &+ \sum_{0<t_i<T} G(t, t_i)(1+Q_i)^{-1}[\mathscr{I}_i(y_{K-1}(t_i)) + \Delta I_i^{K-1}(t_i, h)]\,. \end{aligned} \tag{14}$$

So, we construct the sequence of T-periodic functions $y_K(t)$ $K \leqslant 1$, defined by equalities (14).

II. *Convergence of the subsequent approximations.* Introduce the notations:

$$\begin{array}{llll} y = y(t) & \bar{y} = y(t-h) & \psi = \psi(t) & \bar{\psi} = \psi(t-h) \\ y_i = y(t_i) & \bar{y}_i = y(t_i-h) & \psi_i = \psi(t_i) & \bar{\psi}_i = \psi(t_i-h) \\ x = x(t) & \bar{x}_i = x(t_i-h)\,. & & \end{array}$$

Then

$$
\begin{aligned}
&Y(t,\ y)+\Delta f(t,\ x,\ \bar{x})=\Delta f_0(t,\ h)+g(t)(\bar{y}-y)+q(t)y+S(t)\bar{y}+\overline{Y}(t,\ y,\ \bar{y})\\
(15)\qquad &\\
&\Delta \mathcal{I}_i(y_i)+\Delta I_i(x_i,\ \bar{x}_i)=\Delta I_i^0(t_i,\ h)+Q_i(\bar{y}_i-y_i)+R_i y_i+S_i\bar{y}_i+\tilde{\mathcal{I}}_i(y_i,\ \bar{y}_i)
\end{aligned}
$$

where $\Delta f_0(t,\ h)$, $\Delta I_i^0(t_i,\ h)$ are defined by equalities (11) and

$$
q(t)=\frac{\partial f(t,\ \psi,\ \bar{\psi})}{\partial x}-\frac{\partial f(t,\ \psi,\ \psi)}{\partial x}\qquad s(t)=\frac{\partial f(t,\ \psi,\ \bar{\psi})}{\partial y}-\frac{\partial f(t,\ \psi,\ \psi)}{\partial y}
$$

$$
R_i=\frac{\partial I_i(\psi_i,\ \bar{\psi}_i)}{\partial x}-\frac{\partial I_i(\psi_i,\ \psi_i)}{\partial x}\qquad S_i=\frac{\partial I_i(\psi_i,\ \bar{\psi}_i)}{\partial y}-\frac{\partial I_i(\psi_i,\ \psi_i)}{\partial y}
$$

$$
\begin{aligned}
\overline{Y}(t,\ y,\ \bar{y})=f(t,\ \psi+y,\ \bar{\psi}+\bar{y})-f(t,\ \psi,\ \bar{\psi})-\frac{\partial f(t,\ \psi,\ \bar{\psi})}{\partial x}y&\\
-\frac{\partial f(t,\ \psi,\ \bar{\psi})}{\partial y}\bar{y}-\frac{\partial f(t,\ \psi,\ \psi)}{\partial x}(\bar{y}-y)&
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\mathcal{I}}_i(y_i,\ \bar{y}_i)=I_i(\psi_i+y_i,\ \bar{\psi}_i+\bar{y}_i)-I_i(\psi_i,\ \bar{\psi}_i)-\frac{\partial I_i(\psi_i,\ \bar{\psi}_i)}{\partial x}y_i&\\
-\frac{\partial I_i(\psi_i,\ \bar{c}_i)}{\partial y}\bar{y}_i-\frac{\partial I_i(\psi_i,\ \psi_i)}{\partial x}(\bar{y}_i-y_i)\ .&
\end{aligned}
$$

The approximation $y_K(t)$, $K\geqq 1$, can be written in the form

$$
\begin{aligned}
(16)\qquad y_K(t)=\int_0^T &G(t,\ \tau)[\Delta f_0(\tau,\ h)+g(\tau)(y_{K-1}(\tau-h)-y_{K-1}(\tau))\\
&+q(\tau)y_{K-1}(\tau)+s(\tau)y_{K-1}(\tau-h)+\overline{Y}(\tau,\ y_{K-1}(\tau),\ y_{K-1}(\tau-h)]\,d\tau\\
&+\sum_{0<t_i<T}G(t,\ t_i)[I_i^0(t_i,\ h)+Q_i(y_{K-1}(t_i-h)-y_{K-1}(t_i))\\
&+R_i y_{K-1}(t_i)+S_i y_{K-1}(t_i-h)+\tilde{\mathcal{I}}_i(y_{K-1}(t_i),\ y_{K-1}(t_i-h))]\ .
\end{aligned}
$$

We will employ the following lemma which is a corollary of the theorem about finite increments.

Lemma 1. *Let the function $g(t)$: $R\to R$ be piecewise continuous in $[a, b]$ with first order discontinuity points at $t_i\in(a, b\ )$ $(i=1,\ l)$, $g(t_i+0)-g(t_i-0)=\alpha_i$ and let it be continuously differentiable for $t\neq t_i$.*

Then

$$|g(b\ )-g(a)|\leqq|b\ -a|M+lN$$

where

$$M=\sup_{\substack{\xi\in(a,b)\\ \xi\neq t_i}}(g'(\xi)) \qquad N=\max_{1\leqq i\leqq l}|\alpha_i| .$$

Lemma 2. *Let the following conditions be fulfilled:*

1. *The conditions* (A) *hold.*

2. *The function* $y(t)$ *is a* T*-period solution of the system*

$$\begin{aligned} \dot{y} &= g(t)y+f(t) \qquad & t\neq t_i \\ (\Delta y)_{t=t_i} &= Q_i y(t_i)+\alpha_i \qquad & i\in Z . \end{aligned} \tag{17}$$

Then, the estimate

$$\sup_{t\in[0,T]}|y(t)|\leqq M\max(\max_{[0,T]}|f(t)|,\ \sup_{i\in Z}|(1+Q_i)^{-1}\alpha_i|) \tag{18}$$

holds, where

$$M=\sup_{[0,T]}\Big(\int_0^T|G(t,\ \tau)|\,\mathrm{d}\tau+\sum_{0<t_i<T}|G(t,\ t_i)|\Big) .$$

Proof. The solution of system (17) has the form

$$y(t)=\int_0^T G(t,\ \tau)f(\tau)\,\mathrm{d}\tau+\sum_{0<t_i<T}G(t,\ t_i)(1+Q_i)^{-1}\alpha_i . \tag{19}$$

Equality (19) implies the estimate (18).

Lemma 3. *Let conditions* (A) *be fulfilled.*

Then, a number $\bar{h}>0$ *exists, such that for* $h<\bar{h}$ *the sequence of functions* $\{y_K(t)\}_0^\infty$ *defined by equality* (14), *is uniformly convergent by* t *and its limit is a solution of system* (6).

Proof. Condition A3 implies that the derivatives $\tilde{\mathscr{T}}_{iy}(x, y)$ and $\tilde{\mathscr{T}}_{ix}(x, y)$, $i \in Z$ exist and are continuous. Then we can find a continuous function V_i: $D \times D \to R (i \in Z)$ for which

$$V_i(u, \bar{u}) \geqq |\tilde{\mathscr{T}}_i(y, \bar{y})| \qquad V_{ix}(u, \bar{u}) \geqq |\tilde{\mathscr{T}}_{ix}(y, \bar{y})|$$

$$V_{iy}(u, \bar{u}) \geqq |\tilde{\mathscr{T}}_{iy}(y, \bar{y})|$$

for $u \geqq y$ and $\bar{u} \geqq \bar{y}$, u, $\bar{u}$, y, $\bar{y} \in D$.

Condition A2 implies that the derivatives $\bar{y}_x$ and $\bar{y}_y$ exist and hence we are in a position to find a continuous function U: $D \times D \to R$ for which

$$U(u, \bar{u}) \geqq |\overline{Y}(t, y, \bar{y})| \qquad U_x(u, \bar{u}) \geqq |\overline{Y}_x(t, y, \bar{y})|$$

$$U_y(u, \bar{u}) \geqq |\overline{Y}_y(t, y, \bar{y})|$$

for $u \geqq y$ and $\bar{u} \geqq \bar{y}$, u, $\bar{u}$, y, $\bar{y} \in D$, $t \in [0, T]$.

Define the continuous function W: $D \to R$ for which

$$w(u) \geqq \max (U(u, u) \qquad \sup_{i \in Z} V_i(u, u)) \qquad u \in D .$$

Consider the equations

$$u = \alpha + M[a(hv + pN) + hbu + w(u)]$$

(20)

$$v = \beta + a(u - \alpha) + a(hv + pN) + bhu + w(u)$$

where:

$$\alpha = \sup_{t \in [0, T]} |y_1(t)| \; \beta = \sup_{t \in [0, T]} |\dot{y}_1(t)|$$

$$a \geqq \max (\sup_{t \in [0, T]} |g(t)| \; \sup_{i \in Z} |Q_i|) ,$$

$$bh \geqq \max (\sup_{t \in [0, T]} (|q(t) + s(t)|) \; \sup_{i \in Z} (|R_i| + |S_i|)) .$$

The solution of system (20) is looked for by the method of subsequent approximations. Put

$$u_1 = \alpha \qquad v_1 = \beta$$

$$(21) \qquad \begin{aligned} u_k &= \alpha + M[a(hv_{K-1} + pN) + bhu_{K-1} + w(u_{K-1})] , \\ v_k &= \beta + a(u_K - \alpha) + a[hv_{K-1} + pN] + bhu_{K-1} + w(u_{K-1}) . \end{aligned}$$

Equalities (21) yield that u_k satisfies the equality

$$u_K \leqq \alpha + M \{h [a\beta + (a^2 + \frac{a}{M}) (u_{K-1} - \alpha) + apN + bu_{K-1}] + w(u_{K-1})\} .$$

Denote by $\bar{h}$ the upper bound of the values of h, for which the equation

$$(22) \qquad u = \alpha + M \{h [a\beta + (a^2 + \frac{a}{M}) (u - \alpha) + apN + bu] + w(u)\}$$

has positive solutions. In view of [2] for $h < \bar{h}$ the limit $\lim_{K\to\infty} u_K = u > 0$ exists and the number u is a solution of equation (22). Then, for $h < \bar{h}$, in view of Lemma 1, Lemma 2 and equalities (21) $u_K \geqq |y_K(t)|$, $u_{K+1} - u_K \geqq |y_{K+1}(t) - y_K(t)|$ for $t \in [0, T]$.

Therefore, the sequence of functions $\{y_K(t)\}_0^\infty$ is uniformly convergent by t, and $\lim_{K\to\infty} y_K(t) = y(t)$, where $y(t)$ is a T-periodic solution of system (6).

Lemma 4. *Let conditions* (A) *be fulfilled.*

Then, a number $\bar{\bar{h}} > 0$ *exists, such that for* $h < \bar{\bar{h}}$ *and* $K \geqq 1$, *the relation* $x_K(t) = y_K(t) + \psi(t) \in D$ *holds, for* $t \in [0, T]$.

Proof. For $h = 0$, the solution of system (6) is equal to zero. Hence

$$(23) \qquad \sup_{t\in[0,T]} |y_K(t)| \underset{h\to 0}{\longrightarrow} 0 .$$

Relation (23) implies that a number $\bar{\bar{h}} > 0$ exist, such that for $h < \bar{\bar{h}}$ and $t \in [0, T]$, the function $x_K(t) \in D$ for every $K \geqq 1$.

Theorem 1. *Let conditions* (A) *be fulfilled. Then, a number* $H > 0$ *exists, such that for* $h < H$ *the system* (1) *has a* T*-periodic solution* $x^*(t)$ *which for* $h = 0$ *coincides with* $\psi(t)$.

Proof. Choose $H = \min(\bar{h}, \bar{\bar{h}})$ and construct the sequences of functions $\{y_K(t)\}_0^\infty$, defined by equality (14). In view of Lemma 3, the sequence $\{y_K(t)\}_0^\infty$ is uniformly convergent by t for $h < H$ and its limit $y(t)$ is a solution of system (6). Introduce the denotation $x^*(t) = y(t) + \psi(t)$. The function is T-periodic and satisfies system (1). In view of Lemma 4, for $h = 0$, $y(t) = 0$ and hence $x^*(t) \equiv \psi(t)$.

Thus, Theorem 1 is proved.

References

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[2] J. L. RIABOV, *Determination of the domain of existence of some implicit functions*, C.R. Soviet Energ. Inst., Math. (1957), 12-35 (in Russian).

Summary

In the paper the Poincaré method is employed for finding the periodic solutions of differential-difference equations with impulses, the constant delay being assumed as a small parameter.

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