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Some linear operators connected with abstract differential equations (**)

A Bianca Manfredi per il suo 70° compleanno

Introduction

In this article a linear (unbounded) operator A in the Banach space X is given, and then the first-order inhomogeneous equation: $u'(t) = Au(t) + f(t)$, $t \in]a, b[$.

One of the methods of investigation amounts to consideration of the operator $L = \frac{\mathrm{d}}{\mathrm{d}t} - A$ acting in some space of X-valued functions $u(t)$, as well as of (various) possible extensions of this operator.

Some work of this kind (mostly in Hilbert spaces) has been done previously (see for instance Lions [**4**], Carroll [**1**]); in our paper $[\mathbf{6}]_2$ this method was used, in both Hilbert and general Banach spaces, to get necessary conditions for the regularity of weak solutions. See also $[\mathbf{6}]_1$.

Presently we wish to put a stronger light on some definitions and results in $[\mathbf{6}]_2$, in a $L^p_{\mathrm{loc}}(]a, b[; X)$-setting (where p is any real $p \geq 1$); one should compare with Propositions 5, 6, 7 in $[\mathbf{6}]_2$ which appear to be somewhat less general.

Note that only very simple facts of the linear functional analysis are required as background.

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1 – Consider a Banach space X and an open interval $]a,\ b[$ on the real line. Given the real number $p \geq 1$, the space $L^p_{loc}(]a,\ b[;\ X)$ consists of all functions $u(t)$, $]a,\ b[\rightarrow X$ which are strongly measurable and such that $\int_{a_1}^{b_1} \|u(t)\|_X^p \, dt < \infty$ for any compact subinterval $[a_1,\ b_1]$ of $]a,\ b[$. (See Hille-Phillips [3] or Yosida [5]).

Next, let A be a linear operator with dense domain $D(A) \subset X$, A^* its dual operator, $D(A^*) \subset X^*$-dual space to X. For any interval $]a,\ b[\subset \mathbb{R}$, $K_{A^*}]a,\ b[$ consists of functions $\varphi^*(t)$, $]a,\ b[\rightarrow D(A^*)$, $\varphi^* \in C^1(]a,\ b[;\ X^*)$, supp φ^* is compact in $]a,\ b[$, $A^* \varphi^*$ is continuous, $]a,\ b[\rightarrow X^*$.

Given any function $f \in L^p_{loc}(]a,\ b[;\ X)$, we say that $u \in L^p_{loc}(]a,\ b[;\ X)$ is a (ultra) *weak solution* of the equation

$$u' - Au = f \qquad \text{on} \quad]a,\ b[\tag{1.1}$$

if the integral identity

$$\int_a^b \langle \dot{\varphi}^*(t) + (A^* \varphi^*)(t),\ u(t) \rangle \, dt = - \int_a^b \langle \varphi^*(t),\ f(t) \rangle \, dt \tag{1.2}$$

holds $\forall \varphi^* \in K_{A^*}]a,\ b[$. (Here $\langle , \rangle$ means duality between X and X^*).

Note that in general there is no uniqueness; we can add to a solution u any (strong or weak) solution of the homogeneous equation $u' - Au = \theta$. (See our paper $[6]_1$).

Let $L = \dfrac{d}{dt} - A$ be the linear operator, defined on $D(L) = \{u(t) \in C^1(]a,\ b[;\ X)$, $u(t) \in D(A)\ \forall t \in\ ]a,\ b[$, $Au \in C^0(]a,\ b[;\ X)\}$, by the relation $Lu = \dfrac{du}{dt} - Au$; thus, L maps (linearly) $D(L)$ into $C(]a,\ b[;\ X)$.

Next, let us define a second (linear) operator [1], the weak extension of L, ωL in the following way: $D(\omega L) = \{u \in L^p_{loc}(]a,\ b[;\ X)$, such that $\exists f \in L^p_{loc}(]a,\ b[;\ X)$, with the property that (1.2) above is verified $\forall \varphi^* \in K_{A^*}]a,\ b[)\}$.

Then we put

$$f \in (\omega L)\, u\ . \tag{1.3}$$

Proposition 1. *$D(L) \subset D(\omega L)$ and $Lu \in (\omega L)\, u$ $\forall u \in D(L)$.*

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(1) It could be a multi-valued operator.

Proof. Let be $u \in D(L)$ and then $f = \frac{du}{dt} - Au$, a continuous function, $]a, b[\to X$. Then u and f are in $L^p_{loc}(]a, b[; X)$. Take now any $\varphi^* \in K_{A^*}(]a, b[)$. From the equality

$$u' - Au = f \tag{1.4}$$

we deduce

$$\langle \varphi^*(t), u'(t) \rangle - \langle \varphi^*(t), Au(t) \rangle = \langle \varphi^*(t), f(t) \rangle$$

or also

$$\frac{d}{dt} \langle \varphi^*(t), u(t) \rangle - \langle \dot{\varphi}^*(t), u(t) \rangle - \langle A^* \varphi^*(t), u(t) \rangle = \langle \varphi^*(t), f(t) \rangle;$$

hence the equality

$$\int_a^b \langle \dot{\varphi}^*(t) + A^* \varphi^*(t), u(t) \rangle \, dt = - \int_a^b \langle \varphi^*(t), f(t) \rangle \, dt \tag{1.5}$$

is true; and therefore $f \in (\omega L) u$.

Proposition 2. *The weak extension ωL is a (sequentially) closed operator in $L^p_{loc}(]a, b[; X)$ in the sense that $(u_n) \subset D(\omega L)$, $u_n \to u$ in $L^p_{loc}(]a, b[; X)$ and $(\omega L) u_n$ (2) $\to f$ in $L^p_{loc}(]a, b[; X)$, $\Rightarrow u \in D(\omega L)$ and $f \in (\omega L) u$.*

In fact, we assume the existence of $f_n \in L^p_{loc}(]a, b[; X)$ such that

$$\int_a^b \langle \dot{\varphi}^*(t) + (A^* \varphi^*)(t), u_n(t) \rangle \, dt = - \int_a^b \langle \varphi^*(t), f_n(t) \rangle \, dt \tag{1.6}$$

holds, $\varphi^* \in K_{A^*}]a, b[$ $\forall n = 1, 2, \ldots$, and $f_n \to f$ in $L^p_{loc}(]a, b[; X)$. Then

$$\lim_{n \to \infty} \int_a^b \langle \varphi^*(t), f_n(t) \rangle \, dt = \int_a^b \langle \varphi^*(t), f(t) \rangle \, dt, \qquad \forall \varphi^* \in K_{A^*}]a, b[\tag{1.7}$$

as readily seen.

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(2) This means that $\forall f_n \in (\omega L) u_n$ $f_n \to f$ in the specified sense.

From (1.6), (1.7) and the convergence of u_n towards u in $L^p_{loc}(]a,\ b[;\ X)$-sense we obtain as $n \to \infty$ the identity

$$(1.8) \qquad \int_a^b \langle {}^*(t) + (A^*\varphi^*)(t),\ u(t)\rangle\, dt = -\int_a^b \langle \varphi^*(t),\ f(t)\rangle\, dt \qquad \forall \varphi^* \in K_{A^*}]a,\ b[$$

so that $u \in D(\omega L)$ and $f \in (\omega L)u$.

We have now the following

Proposition 3. *Let A be also a closed linear operator in X (so that $D(A^*)$ is a total set in X^*). Then the weak extension ωL is a single-valued mapping.*

Proof. Assume that f_1, f_2 both belong to $(\omega L)u$; we get readily from (1.2) the equality

$$(1.9) \qquad \int_a^b \langle \varphi^*(t),\ f_1(t) - f_2(t)\rangle\, dt = 0 \qquad \forall \varphi^* \in K_{A^*}]a,\ b[\ .$$

In particular, for $\varphi^*(t) = \psi(t)x^*$, where $\psi \in C_0^1]a,\ b[$ and $x^* \in D(A^*)$, we infer that $\langle x^*, \int_a^b \psi(t)(f_1(t) - f_2(t))\, dt\rangle = 0$. Thus ($D(A^*)$ is total!), it follows that

$$(1.10) \qquad \int_a^b \psi(t)(f_1(t) - f_2(t))\, dt = \theta \qquad \forall \psi \in C_0^1]a,\ b[\ .$$

Now, take $t_0 \in]a,\ b[$ and a small $\tau > 0$; define

$$\psi_{t_0}(t) = \begin{cases} \dfrac{1}{\tau} & t_0 \le t \le t_0 + \tau \\ 0 & t \notin [t_0,\ t_0 + \tau] \end{cases}$$

By convolution with a δ-like sequence $\{\alpha_n\}$, we obtain $\psi_{t_0,n} = \psi_{t_0} * \alpha_n \in C_0^\infty]a,\ b[$ (for n sufficiently large) and $\psi_{t_0,n} \to \psi_{t_0}$ in $L^{p'}(\mathrm{R})$ sense $(1/p + 1/p' = 1)$.

Therefore

$$(1.11) \qquad \int_a^b \psi_{t_0}(t)(f_1(t) - f_2(t))\, dt = \lim_{n\to\infty} \int_a^b \psi_{t_0,n}(t)(f_1(t) - f_2(t))\, dt = \theta\ .$$

We get

$$\frac{1}{\tau}\int_{t_0}^{t_0+\tau} [f_1(t) - f_2(t)]\, dt = \theta \qquad \text{for small} \quad \tau > 0$$

(note that $f_1 - f_2$ is Bochner-integrable in $[t_0,\ t_0 + \lambda] \subset]a,\ b[$).

As $\tau \to 0$ one obtains $f_1(t_0) = f_2(t_0)$ a.e. in $]a, b[$. (Compare with Propositions 1 and 5 in the work [6]$_2$).

Remark. Assume that A is not closed and that $D(A^*) = \{\theta\}$ (see for instance [2], p. 53, Ex. II.2.7).

It follows that $K_{A^*}]a, b[$ reduces to the constant null function $\varphi^*(t) = \theta$ $\forall t \in]a, b[$. Accordingly, (1.2) holds true, given any $u \in L^p_{loc}(]a, b[; X)$ and any $f \in L^p_{loc}(]a, b[; X)$. Therefore $D(\omega L) = L^p_{loc}(]a, b[; X)$ and any $f \in L^p_{loc}(]a, b[; X)$ belongs to $(\omega L)u$.

We end with a final result establishing the linearity of the (multi-valued) operator ωL when A is not necessarily closed. Precisely, we have the (3)

Proposition 4. *If $u_1, u_2 \in D(\omega L)$ and $\lambda \in \mathbf{C}$, then $u_1 + u_2$ and λu_1 belong to $D(\omega L)$. Also, if $f_1 \in (\omega L) u_1$ and $f_2 \in (\omega L) u_2$, it folllows that $f_1 + f_2 \in (\omega L)(u_1 + u_2)$ and $\lambda f_1 \in (\omega L)(\lambda u_1)$.*

Proof. In fact, we assume the existence of $f_1, f_2 \in L^p_{loc}(]a, b[; X)$ such that

$$(1.12) \qquad \int_a^b \langle \dot{\varphi}^* + A^* \varphi^*, u_i \rangle \, dt = - \int_a^b \langle \varphi^*(t), f_i(t) \rangle \, dt \qquad i = 1, 2, \ \forall \varphi^* \in K_{A^*}]a, b[\, .$$

Summing one gets

$$(1.13) \qquad \int_a^b \langle \dot{\varphi}^* + A^* \varphi^*, u_1 + u_2 \rangle \, dt = - \int_a^b \langle \varphi^*(t), f_1 + f_2 \rangle \, dt \qquad \forall \varphi^* \in K_{A^*}]a, b[\, .$$

This means: $u_1 + u_2 \in D(\omega L)$ and $f_1 + f_2 \in (\omega L)(u_1 + u_2)$.

Similarly, we note that

$$\int_a^b \langle \dot{\varphi}^* + A^* \varphi^*, \lambda u_1 \rangle \, dt = \lambda \int_a^b \langle \dot{\varphi}^* + A^* \varphi^*, u_1 \rangle \, dt = - \int_a^b \langle \varphi^*(t), \lambda f_1(t) \rangle \, dt \, .$$

Thus $\lambda u_1 \in D(\omega L)$ and $\lambda f_1 \in (\omega L)(\lambda u_1)$.

(3) See Propositions 2 and 6 in [6]$_2$.

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Abstract

We consider linear inhomogeneous differential equations in Banach spaces X of the form $\frac{du}{dt} - Au = f$, *in an open interval* $]a, b[\subset \mathbb{R}$ *and then define and investigate a certain weak extension of the operator* $\frac{d}{dt} - A$ *considered in the space* $L^p_{loc}(]a, b[; X)$, $p \geq 1$.

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