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Universal solutions for elastic shells (**)

1 - Introduction

The problem of static universal solutions for thin two-dimensional continuum media, finitely deformed, has been discussed by several authors (see for example [2], [5], ..., [9] and their references).

We recall that the equilibrium solutions are universal if they are equilibrium configurations for every isotropic elastic material.

We already studied in [1] the problem of universal solutions in the case of the linear restricted theory for shells.

The aim of the present paper is to study the static universal solutions for every elastic shell initially homogeneous and isotropic within the framework of a direct theory, both in the non-linear and in the linear case.

Our first purpose is to determine and to analyse the set of conditions which restrict the class of universal solutions for elastic membranes (see [5]) to the class of universal solutions for elastic shells in the non-linear theory. As we already did in [1], we refer essentially to the procedure used in [5] by Naghdi and Tang in the case of finitely deformed elastic membranes.

Secondly we extend the results already get in [1] to the case of the linear complete theory for shells.

In 2a we recall briefly the geometric and kinematic variables for a shell in the non-linear restricted theory as well as the equilibrium equations and the constitutive relations appropriate for the direct theory, as proposed in [4] and [3]

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(n. 7). In 2b we specify the assigned force and we give the explicit conditions to be imposed in order to obtain all universal solutions. Since one set of conditions is the same as one given in [5], in 2c we make use of the solutions for the two fundamental forms of the deformed surface given by Naghdi and Tang in [5]. We observe that the only universal deformations are from planes or right circular cylinders into planes or right circular cylinders and from spheres into spheres. In 3 we examine the case of infinitesimal deformations in the complete theory. We refer to [1] for the procedure in order to find explicit solutions and we observe that the only possible deformations are: a plane into a plane, a right circular cylinder into a right circular cylinder, a sphere into a sphere. Finally we point out that deformations along the director are possible.

2 - The non-linear restricted theory

2a. *Basic equations*

Let us consider a material surface s embedded in a Euclidean 3-space and identify the material points of the surface with convected coordinates θ^α $(\alpha = 1,\ 2)$. We denote the position vector of s with $\boldsymbol{r} = \boldsymbol{r}(\theta^\alpha,\ t)$, where t is time; let $\boldsymbol{a}_\alpha \equiv \boldsymbol{r}_{,\alpha}$ be the base vectors along the θ^α-curves on s and $\boldsymbol{a}_3 = \dfrac{\boldsymbol{a}_1 \times \boldsymbol{a}_2}{|\boldsymbol{a}_1 \times \boldsymbol{a}_2|}$ the unit normal to s. In the above formula and throughout the paper a comma stands for partial differentiation with respect to θ^α. Let us remember the following relations

$$\boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta = a_{\alpha\beta} \qquad a = \det(a_{\alpha\beta}) > 0 \qquad \boldsymbol{a}^\alpha \cdot \boldsymbol{a}_\beta = \delta^\alpha_\beta \qquad \boldsymbol{a}^\alpha \cdot \boldsymbol{a}^\beta = a^{\alpha\beta}$$

$$a^{\alpha\gamma} a_{\gamma\beta} = \delta^\alpha_\beta \qquad \boldsymbol{a}^\alpha = a^{\alpha\gamma} \boldsymbol{a}_\gamma \qquad \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_3 = 0 \qquad \boldsymbol{a}^3 = \boldsymbol{a}_3\ .$$

In the above relations $a_{\alpha\beta}$ and $a^{\alpha\beta}$ are the components of the first fundamental form of s and its conjugate, δ^α_β is the Kronecker symbol in 2-space. The surface s with a normal vector field $\boldsymbol{a}_3$ represents a shell in the restricted theory and its motion is characterized by

$$\boldsymbol{r} = \boldsymbol{r}(\theta^\alpha,\ t) \qquad \boldsymbol{a}_3 = \boldsymbol{a}_3(\theta^\alpha,\ t)\ .$$

Let $b_{\alpha\beta}$ denote the components of the second fundamental form of s

$$b_{\alpha\beta} = - \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_{3,\beta} = \boldsymbol{a}_3 \cdot \boldsymbol{a}_{\alpha,\beta} .$$

Then the mean and Gaussian curvature of s are, respectively

$$H = \frac{1}{2} b_{\alpha\beta} a^{\alpha\beta} \qquad K = \frac{\det(b_{\alpha\beta})}{\det(a_{\alpha\beta})} . \tag{2.1}$$

We also recall the formulae of Mainardi-Codazzi and Gauss

$$b_{\alpha\beta|\gamma} = b_{\alpha\gamma|\beta} \qquad R_{1212} = K \det(a_{\alpha\beta}) \tag{2.2}$$

where a vertical bar (|) stands for covariant differentiation with respect to the metric tensor $a_{\alpha\beta}$ and R_{1212} is the only independent component of the covariant surface curvature tensor.

Let $\boldsymbol{R} = \boldsymbol{R}(\theta^\alpha)$ denote the position vector of the (initial) undeformed configuration S of the shell. In the following we use capital letters to represent the duals of quantities associated with s in the reference surface S. The base vectors, the unit normal vector, the components of the first and second fundamental form, the mean and Gaussian curvature of S will be denoted by $\boldsymbol{A}_\alpha$, $\boldsymbol{A}_3$, $A_{\alpha\beta}$, $B_{\alpha\beta}$, $\bar{H}$, $\bar{K}$. Then the formulae of Mainardi-Codazzi and Gauss for S are

$$B_{\alpha\beta\|\gamma} = B_{\alpha\gamma\|\beta} \qquad \bar{R}_{1212} = \bar{K} A \tag{2.3}$$

where $\bar{R}_{1212}$ is the only independent component of the covariant surface curvature tensor and A stands for $\det(A_{\alpha\beta})$.

In $(2.3)_1$ and in the following the double bars ($\|$) stand for covariant differentiation with respect to the metric tensor $A_{\alpha\beta}$.

We now introduce the kinematic variables in the case of the nonlinear deformations

$$e_{\alpha\beta} = \frac{1}{2}(a_{\alpha\beta} - A_{\alpha\beta}) = e_{\beta\alpha} \qquad \varkappa_{\alpha\beta} = B_{\alpha\beta} - b_{\alpha\beta} = \varkappa_{\beta\alpha} . \tag{2.4}$$

The equilibrium equations in the absence of body force are

$$\begin{aligned} &\grave{N}^{\alpha\beta}{}_{|\alpha} - b_\gamma{}^\beta{}_{|\alpha} \grave{M}^{(\gamma\alpha)} - 2 b_\alpha{}^\beta \grave{M}^{(\gamma\alpha)}{}_{|\gamma} + \rho F^\beta = 0 \\ &\grave{M}^{(\alpha\beta)}{}_{|\alpha\beta} + b_{\alpha\beta} \grave{N}^{\alpha\beta} - b_{\alpha\beta} b_\gamma{}^\beta \grave{M}^{(\gamma\alpha)} + \rho F^3 = 0 \end{aligned} \tag{2.5}$$

where ρ denotes the mass density of the surfaces s, F^{β} and F^{3} are the components of the assigned surface force $\boldsymbol{F}$ for unit mass, referred to the base vectors $(\boldsymbol{a}_{\alpha}, \boldsymbol{a}_{3})$. The above equations correspond to (15.20) (p. 552) of [4] in the static case; for the boundary conditions appropriate to these theory see [4] ((15.24), (15.25), p. 553).

The local equation for conservation of mass can be expressed as

$$\rho a^{\frac{1}{2}} = \rho_0 A^{\frac{1}{2}} \tag{2.6}$$

where ρ_0 denotes the mass density of the reference surface S.

Restricting attention here to the isothermal theory of elastic shells, we assume that the strain energy density φ does not depend explicitly on $B_{\alpha\beta}$ (see [5]). Therefore φ takes the form

$$\varphi = \varphi(e_{\alpha\beta},\ x_{\alpha\beta}) \ .$$

When the shell is initially isotropic with a center of symmetry, φ may be expressed as a function of the following joint invariants (see [3], (7.25))

$$\begin{gathered} I_1 = A^{\alpha\beta} e_{\alpha\beta} \qquad I_2 = A^{\alpha\gamma} A^{\beta\delta} e_{\alpha\beta} e_{\gamma\delta} \qquad I_3 = A^{\alpha\gamma} A^{\beta\delta} e_{\alpha\beta} x_{\gamma\delta} \\ I_4 = A^{\alpha\beta} x_{\alpha\beta} \qquad I_5 = A^{\alpha\gamma} A^{\beta\delta} x_{\alpha\beta} x_{\gamma\delta} \ . \end{gathered} \tag{2.7}$$

Therefore

$$\varphi = \hat{\varphi}(I_1,\ I_2,\ I_3,\ I_4,\ I_5) \ . \tag{2.8}$$

The constitutive equations for $\grave{N}^{\alpha\beta}$ and $\grave{M}^{(\alpha\beta)}$ are

$$\grave{N}^{\alpha\beta} = \rho \frac{\partial \varphi}{\partial e_{\alpha\beta}} = \rho \sum_{1}^{3}{}_r \varphi_r \frac{\partial I_r}{\partial e_{\alpha\beta}} \qquad \grave{M}^{(\alpha\beta)} = \rho \frac{\partial \varphi}{\partial x_{\alpha\beta}} = \rho \sum_{3}^{5}{}_r \varphi_r \frac{\partial I_r}{\partial x_{\alpha\beta}} \tag{2.9}$$

where
$$\varphi_s = \frac{\partial \varphi}{\partial I_s} \ , \qquad s = 1,\ \ldots,\ 5.$$

2b. *Formulation of the problem*

Let us now study the static universal solutions for every isotropic elastic shell in the framework of a non-linear restricted theory. We assume that the strain energy density (2.8) depends on each argument in distinct way. We suppose the initial mass density ρ_0 uniform across S, i.e.

(2.10) $$\rho_0 = \text{const} .$$

Let the components of the assigned surface force be

(2.11) $$F^\beta = 0 \qquad F^3 = \text{const} .$$

We assume, moreover, that the components of the first fundamental form, both of the initial surface and the deformed one, are tensor functions of class $C^{(2)}$, while the components of the second fundamental form are tensor functions of class $C^{(1)}$.

If the set of variables $(A_{\alpha\beta}, B_{\alpha\beta})$ of the reference surface S and the deformation variables $(e_{\alpha\beta}, x_{\alpha\beta})$ describes the static universal deformations resulting from the assigned surface force (2.11), as well as appropriate edge loads, then $(A_{\alpha\beta}, B_{\alpha\beta})$, $(e_{\alpha\beta}, x_{\alpha\beta})$ must satisfy:

the continuity equation (2.6) with $\rho_0 = \text{const}$,
the equilibrium equations (2.5),
the compatibility equations (2.2), (2.3)

for every choice of the response function $\hat{\varphi}$ in (2.8).

Let us consider now $(2.5)_1$ in which $\grave{N}^{\alpha\beta}$ and $\grave{M}^{(\alpha\beta)}$ are given by (2.9). In order that $(2.5)_1$ be satisfied by the variables $A_{\alpha\beta}$, $B_{\alpha\beta}$, $e_{\alpha\beta}$ and $x_{\alpha\beta}$ for every choice of $\hat{\varphi}$, it is necessary and sufficient that the coefficients of each distinct derivative of $\hat{\varphi}$ in $(2.5)_1$ vanish independently. Equating to zero these coefficients we get a set of restrictions which we will specify afterwards. We restrict now our attention to one of the conditions of this set

(2.12) $$A^{\alpha\beta} I_{s,\alpha} = 0 .$$

Since $(A^{\alpha\beta})$ is non-singular, (2.12) yields

(2.13) $$I_s = \text{const} \qquad s = 1, \ldots, 5 .$$

In view of the relation (see [5], (2.14))

$$\frac{a}{A} = \frac{1}{2}(\bar{I}_1^2 - \bar{I}_2) \quad \text{with} \quad \bar{I}_1 = A^{\alpha\beta} a_{\alpha\beta} \qquad \bar{I}_2 = A^{\alpha\gamma} A^{\beta\delta} a_{\alpha\beta} a_{\gamma\delta}$$

by means of (2.6), (2.10) and (2.13), we deduce

(2.14) $$\rho = \text{const} .$$

Let us turn now to $(2.5)_2$ in which $\grave{N}^{\alpha\beta}$ and $\grave{M}^{(\alpha\beta)}$ are given by (2.9). We differentiate $(2.5)_2$ with respect to θ^α and we use $(2.11)_2$, (2.13), (2.14); we proceed in the same way as for $(2.5)_1$ and we get a second set of restrictions. Then, with the use of (2.4), (2.7), the mentioned restrictions imposed by $(2.5)_{1,2}$ yield the following conditions:

$$\begin{gathered}\bar{I}_1 = c_1 \qquad \bar{I}_2 = c_2 \qquad A^{\alpha\beta} b_{\alpha\beta} = c_3 \qquad A^{\alpha\gamma} A^{\beta\delta} a_{\gamma\delta} b_{\alpha\beta} = c_4 \\ (\sqrt{A/a}\, A^{\alpha\beta})_{|\alpha} = 0 \qquad (\sqrt{A/a}\, A^{\alpha\gamma} A^{\beta\delta} a_{\gamma\delta})_{|\alpha} = 0\end{gathered} \tag{2.15}$$

and

$$\begin{gathered}\bar{H} = A^{\alpha\beta} B_{\alpha\beta} = \text{const} \qquad B^{\alpha\beta} b_{\alpha\beta} - A^{\alpha\gamma} A^{\beta\delta} b_{\gamma\delta} b_{\alpha\beta} = \text{const} \\ B_{\alpha\beta} B^{\alpha\beta} - 2B^{\gamma\delta} b_{\gamma\delta} + A^{\alpha\gamma} A^{\beta\delta} b_{\alpha\beta} b_{\gamma\delta} = \text{const} \qquad B^{\alpha\beta} a_{\alpha\beta} = \text{const} \\ A^{\alpha\gamma} b_\gamma^{\ \beta} b_{\alpha\beta} = \text{const} \qquad B^{\alpha\gamma} b_\gamma^{\ \beta} b_{\alpha\beta} - A^{\alpha\rho} A^{\gamma\sigma} b_{\rho\sigma} b_\gamma^{\ \beta} b_{\alpha\beta} = \text{const} \\ A^{\alpha\rho} A^{\gamma\sigma} a_{\rho\sigma} b_\gamma^{\ \beta} b_{\alpha\beta} = \text{const} \qquad B^{\alpha\beta}{}_{|\alpha} - (A^{\alpha\gamma} A^{\beta\delta} b_{\gamma\delta})_{|\alpha} = 0 \\ A^{\alpha\gamma} b_\gamma{}^\beta{}_{|\alpha} = 0 \qquad B^{\alpha\gamma} b_\gamma{}^\beta{}_{|\alpha} - A^{\alpha\rho} A^{\gamma\sigma} b_{\rho\sigma} b_\gamma{}^\beta{}_{|\alpha} = 0 \\ A^{\alpha\rho} A^{\gamma\sigma} a_{\rho\sigma} b_\gamma{}^\beta{}_{|\alpha} = 0 \qquad A^{\alpha\beta}{}_{|\alpha\beta} = \text{const} \\ B^{\alpha\beta}{}_{|\alpha\beta} - A^{\alpha\gamma} A^{\beta\delta} b_{\gamma\delta|\alpha\beta} = \text{const} \qquad A^{\alpha\gamma} A^{\beta\delta} a_{\gamma\delta|\alpha\beta} = \text{const}\,.\end{gathered} \tag{2.16}$$

2c. *Universal solutions*

The conditions (2.15) are the same as (4.8) in [5]. The system of equations $(2.15)_{1,2,5,6}$ yields the solution for the first fundamental form in two different cases, according to whether or not the Gaussian curvature $\bar{K}$ is zero. The remaining equations $(2.15)_{3,4}$ yield the solution for the second fundamental form in the cases $\bar{K} = 0$ and $\bar{K} \gtreqless \dfrac{c_3^{\ 2}}{2c_1}$. We refer to [5] (sect. 5) for the explicit form of these solutions; we must nevertheless verify if the solutions for $a_{\alpha\beta}$ and $b_{\alpha\beta}$ satisfy the conditions (2.16).

(i) $\bar{K}=0$. In this case condition $(2.16)_1$ implies that the undeformed surface S must be a plane or a right circular cylinder (see [5], Theorems 6.1.-6.3.). The solutions for the components of the first and second fundamental forms yield therefore (see [5], formulae (5.17), (5.27))

$$(2.17) \qquad a_{\alpha\beta}=\text{const} \qquad b_{\alpha\beta}=\text{const}\,.$$

From $(2.17)_1$ we deduce $R_{1212}=0$ and by $(2.2)_2$ $K=0$; from (2.1) with the use of (2.17) we get $H=\text{const}$. The deformed surface s must therefore be a plane or a right circular cylinder. By observing that in this case the covariant differentiation on s is reduced to the ordinary partial differentiation, the remaining conditions (2.16) are verified.

(ii) $\bar{K}=\dfrac{c_3^{\,2}}{2c_1}>0$. In this case, S is a spherical surface; condition $(2.16)_1$ is satisfied and from (5.7), (5.26) of [5] we get

$$(2.18) \qquad a_{\alpha\beta}=\frac{c_1}{c_3}\,b_{\alpha\beta}\,.$$

With the use of (2.18), formula $(2.1)_2$ gives $K=(\dfrac{c_3}{c_1})^2>0$, that is the deformed surface s is a sphere. The remaining conditions (2.16) are satisfied.

(iii) $\bar{K}<\dfrac{c_3^{\,2}}{2c_1}$. If $\bar{K}=\text{const}$, the integrability condition (5.25) of [5] yields $\bar{K}=0$ (a case already discussed above).

If $\bar{K}\neq\text{const}$, by substituting the solution for $b_{\alpha\beta}$, the conditions $(2.16)_{1,2}$ and the identity $B_{\alpha\beta}B^{\alpha\beta}=4\bar{H}^2-2\bar{K}$ in $(2.16)_3$, we obtain $\bar{K}=\text{const}$ which is not consistent with the hypothesis.

We observe here that the only families of universal deformations for an isotropic elastic shell in the framework of the non linear restricted theory are the families considered in sect. 8 of [5], that is:

family 1 - S and s are both planes;
family 2 - S is a sector of a right circular cylinder and s is a plane;
family 3 - S is a plane and s is a sector of a right circular cylinder;
family 4 - S and s are both right circular cylinders (or sectors of these surfaces);
family 5 - S and s are both spheres (or sectors of these surfaces).

We refer to sect. 8 of [5] for a detailed discussion of the possible deformations of S.

3 - The complete theory

Let s be a material surface representing the shell in the deformed configuration. We assigne to every point of s a deformable vector $\boldsymbol{d}$, called director, which is not necessarily along the unit normal $\boldsymbol{a}_3$ to s. The motion of the shell in the complete theory is characterized by

$$\text{(3.1)} \qquad \boldsymbol{r} = \boldsymbol{r}(\theta^\alpha,\ t) \qquad \boldsymbol{d} = \boldsymbol{d}(\theta^\alpha,\ t) \qquad \boldsymbol{a}_1 \times \boldsymbol{a}_2 \cdot \boldsymbol{d} > 0$$

with $\boldsymbol{a}_\alpha$ base tangent vectors on s.

We refer to [2], (sect. 2) and [3], (sect. 4-5) for the basic equations of the non-linear isothermal shell theory. If the elastic shell is initially isotropic with a center of symmetry, the strain energy density can be expressed as a function of twenty-five joint invariants associated to the deformation variables.

We could study the static universal solutions in this case by applying the same procedure as in the previous sections; nevertheless we leave this problem for future works because of the too high number of conditions involved.

Let us therefore study the problem of static universal solutions when the deformations of the shell characterized by (3.1) are infinitesimal.

3a. *Basic equations of the linear theory*

If $\boldsymbol{R} = \boldsymbol{R}(\theta^\alpha)$ and $\boldsymbol{D} = \boldsymbol{A}_3$ denote respectively the position vector and the normal field vector in the undeformed configuration S of the shell, the displacement $\boldsymbol{u}$ and the director displacement $\boldsymbol{\delta}$ are given by (see [4], (6.1), (6.2), p. 456)

$$\text{(3.2)} \qquad \boldsymbol{u} = \boldsymbol{r} - \boldsymbol{R} \qquad \boldsymbol{u} = u^i \boldsymbol{A}_i = u_i \boldsymbol{A}^i \qquad \boldsymbol{\delta} = \boldsymbol{d} - \boldsymbol{D} \qquad \boldsymbol{\delta} = \delta_i \boldsymbol{A}^i \,.$$

We suppose that the components of $\boldsymbol{u}$ and $\boldsymbol{\delta}$ and their derivatives are small so that we may neglect squares and products of these quantities compared with their first powers.

The linear isothermal deformations of the shell are characterized by the kinematic variables (see [4], (5.31)-(5.34), p. 452)

$$\text{(3.3)} \qquad \begin{aligned} & e_{\alpha\beta} = \frac{1}{2}(a_{\alpha\beta} - A_{\alpha\beta}) \qquad && \varkappa_{\alpha\beta} = \lambda_{\alpha\beta} + B_{\alpha\beta} \qquad && \varkappa_{3\alpha} = \lambda_{3\alpha} \\ & \gamma_\alpha = d_\alpha && \gamma_3 = d_3 - 1 \,. \end{aligned}$$

The expressions for the kinematical quantities introduced in (3.3) can be found in [4], (sect. 6). The compatibility equations which are necessary and sufficient conditions for the existence of single-valued displacement $\boldsymbol{u}$ and $\boldsymbol{\delta}$ are given in [4] (formulae (6.60), (6.61), pp. 465-466).

The equilibrium equations, in the absence of body force are (see [4], (9.69), (9.70), p. 501)

$$(3.4)\qquad \begin{array}{ll} N^{\alpha\beta}{}_{\|\alpha} - B^{\beta}_{\alpha} V^{\alpha} + \rho_0 F^{\beta} = 0 & V^{\alpha}{}_{\|\alpha} + B_{\alpha\beta} N^{\alpha\beta} + \rho_0 F^{3} = 0 \\ M^{\alpha\beta}{}_{\|\alpha} + \rho_0 L^{\beta} = V^{\beta} & M^{\alpha 3}{}_{\|\alpha} + \rho_0 L^{3} = V^{3} \end{array}$$

where ρ_0 is the constant mass density on S, while F^i and L^i are the components, referred to the base vectors $\boldsymbol{A}_i$ of the assigned surface force and of the surface director force per unit mass, respectively.

In the linear theory the local equation for conservation of mass can be expressed as (see [4], (6.14), p. 457)

$$(3.5)\qquad \rho = \rho_0(1 - e^{\alpha}_{\alpha})$$

ρ being the mass density on s.

Restricting attention here to isothermal deformations, the strain energy density ψ for an initially homogeneous isotropic elastic shell with a center of symmetry assumes the form (see [4], (16.21), p. 557)

$$(3.6)\qquad \psi = \frac{1}{\rho_0}\Big\{\frac{1}{2}[\alpha_1 I_1^2 + \alpha_3 I_8 + \alpha_4 I_9^2 + \alpha_5 I_5^2 + (\alpha_6 + \alpha_7) I_4 + \alpha_8 I_6] + \alpha_2 I_2 + \alpha_9 I_1 I_9 + \alpha_{10} I_1 I_5 + 2\alpha_{11} I_3 + \alpha_{12} I_5 I_9 + \alpha_{13} I_7\Big\}\ .$$

The coefficients α_s ($s = 1, \dots, 13$) are constant (see [3], p. 303). They are moreover arbitrary, within the framework of the direct theory (see [4], p. 598). The joint invariants I_r ($r = 1, \dots, 9$) are defined as follows

$$(3.7)\qquad \begin{array}{lllll} I_1 = A^{\alpha\beta} e_{\alpha\beta} & I_2 = A^{\alpha\gamma} A^{\beta\delta} e_{\alpha\beta} e_{\gamma\delta} & I_3 = A^{\alpha\gamma} A^{\beta\delta} e_{\alpha\beta} x_{\gamma\delta} & I_4 = A^{\alpha\gamma} A^{\beta\delta} x_{\alpha\beta} x_{\gamma\delta} & \\ I_5 = A^{\alpha\beta} x_{\alpha\beta} & I_6 = A^{\alpha\beta} x_{3\alpha} x_{3\beta} & I_7 = A^{\alpha\beta} \gamma_{\alpha} x_{3\beta} & I_8 = A^{\alpha\beta} \gamma_{\alpha} \gamma_{\beta} & I_9 = \gamma_3\ . \end{array}$$

In the equilibrium equations (3.4) we set

$$(3.8)\qquad V^{\alpha} = N^{\alpha 3} = m^{\alpha} + B_{\gamma}{}^{\alpha} M^{\gamma 3} \qquad V^3 = m^3 - B_{\alpha\beta} M^{\alpha\beta} \qquad N^{\alpha\beta} = \dot{N}^{\alpha\beta} - M^{\gamma\alpha} B_{\gamma}{}^{\beta}\ .$$

Then the constitutive equations which must be added to (3.4) are (see [4], (16.7), p. 554)

$$\grave{N}^{\alpha\beta} = \grave{N}^{\beta\alpha} = \rho_0 \frac{\partial \psi}{\partial e_{\alpha\beta}} \qquad m^i = \rho_0 \frac{\partial \psi}{\partial \gamma_i} \qquad M^{\alpha i} = \rho_0 \frac{\partial \psi}{\partial x_{i\alpha}} . \tag{3.9}$$

3b. *Formulation of the problem*

In order to study the static universal solutions for every linear elastic shell, initially homogeneous and isotropic, whose strain energy density is specified by (3.6), let us now suppose that the components of the assigned surface force density and of the surface director force density are, respectively

$$F^{\beta} = 0 \qquad F^3 = \text{const} \qquad L^{\beta} = 0 \qquad L^3 = \text{const} . \tag{3.10}$$

We assume, moreover, that the components of the first and second fundamental form of the undeformed surface S and the set of deformation variables (3.3) are sufficiently smooth.

If the set of the variables $(A_{\alpha\beta},\ B_{\alpha\beta})$ of the reference surface S and of the deformation variables $(e_{\alpha\beta},\ x_{i\alpha},\ \gamma_i)$ describes the static universal deformations resulting from the assigned force (3.10) as well as appropriate edge loads, then $(A_{\alpha\beta},\ B_{\alpha\beta})$, $(e_{\alpha\beta},\ x_{i\alpha},\ \gamma_i)$ must satisfy:

the continuity equation (3.5),
the equilibrium equations (3.4),
the compatibility equations (2.3) and (6.60)-(6.61) in [4],

for every choice of the response function ψ in (3.6).

Let us remark that this requirement about the response function ψ is equivalent to assuming that a deformation is possible for all arbitrary values of the coefficients α_1 to α_{13} in (3.6).

By means of (3.6)-(3.9) and by observing that $F^3 = \text{const}$ and $L^3 = \text{const}$ if and only if $F^3{}_{,\sigma} = 0$ and $L^3{}_{,\sigma} = 0$, we obtain from (3.4) a set of non trivial restrictions on the variables $A_{\alpha\beta}$, $B_{\alpha\beta}$, $e_{\alpha\beta}$, $x_{i\alpha}$, γ_i (see 2b). This set of restrictions, with the use of (3.3), becomes

$$\begin{aligned} &\bar{H} = \text{const} \qquad \bar{K} = \text{const} \qquad \gamma_3 = \text{const} \qquad \gamma_\alpha = 0 \\ &\bar{I}_1 \equiv A^{\alpha\beta} a_{\alpha\beta} = \text{const} \qquad H = \text{const} \qquad A^{\alpha\gamma} A^{\beta\delta} a_{\gamma\delta\|\alpha} = 0 \\ &A^{\alpha\gamma} A^{\beta\delta} b_{\gamma\delta\|\alpha} = 0 \qquad B^{\alpha\beta} a_{\alpha\beta} = \text{const} \qquad B^{\alpha\beta} b_{\alpha\beta} = \text{const} \\ &B^{\alpha}_{\beta} B^{\gamma\beta} a_{\gamma\alpha} = \text{const} \qquad B^{\alpha}_{\beta} B^{\gamma\beta} B_{\gamma\alpha}(1-\gamma_3) - B^{\alpha}_{\beta} B^{\gamma\beta} b_{\gamma\alpha} = \text{const} . \end{aligned} \tag{3.11}$$

In (3.11), we have taken in account that $x_{\alpha\beta} = B_{\alpha\beta}(1 - \gamma_3) - b_{\alpha\beta}$, because is $\gamma_3 = \text{const}$.

3c. *Universal solutions*

In order to study the problem of universal solutions, we can proceed exactly as in the restricted linear theory and we refer to [1] for the explicit calculations.

As in the linear restricted theory, we remark that the only families of universal deformations are:

family 1 - S and s are both planes;
family 2 - S and s are both right circular cylinders (or sectors of these surfaces);
family 3 - S and s are both spheres (or sectors of these surfaces).

From $(3.2)_3$ we get the expression for the deformable director relative to every point of s in each of the above families

$$d = A_3 + \delta \,. \tag{3.12}$$

With the use of $(3.3)_{4,5}$ the components δ_i of $\boldsymbol{\delta}$ in $(3.2)_4$ are given by

$$\delta_\alpha = \gamma_\alpha + \beta_\alpha \qquad \delta_3 = \gamma_3 \,. \tag{3.13}$$

We can deduce that β_α ($\alpha = 1, 2$) vanishe for the universal deformations characterized by families 1-3, therefore $(3.11)_4$, (3.12), (3.13) yield

$$\boldsymbol{d} = (1 + \gamma_3)\boldsymbol{A}^3 \,.$$

The condition $(3.1)_3$ or equivalently $d_3 > 0$ implies $\gamma_3 > -1$.

Let us observe that in the linear theory a plane cannot deform itself into a sector of a cylinder or viceversa; moreover the inversion both of the cylinder and the sphere is not consistent with the linear theory (see [1], sects. 3, 4).

We could finally calculate the expressions for the components $N^{\alpha\beta}$, V^α, V^3, $M^{\alpha i}$, F^3 and L^3 as in [1] (sect. 5) but we omit their final expressions since the calculations are straightforward.

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Sommario

Nell'ambito della teoria diretta delle volte alla Cosserat, vengono studiate le soluzioni statiche universali per deformazioni isoterme di una volta elastica isotropa, sia nel caso della teoria ristretta non lineare che in quello della teoria lineare.

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