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On the absolute summability factors for infinite series (**)

1 - Introduction

Let $\sum a_n$ be a given infinte series with the sequence of partial sums (s_n). Let (p_n) be a sequence of real positive constants such that

(1.1) $$P_n = \sum_{v=0}^{n} p_v \to \infty \quad \text{as} \quad n \to \infty \quad (P_{-k} = p_{-k} = 0, \ k \geqslant 1) .$$

The sequence-to-sequence transformation

(1.2) $$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \quad (P_n \neq 0)$$

defines the sequence (t_n) of the $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the sequences of coefficients (p_n). The series $\sum a_n$ is said to be *summable* $|\bar{N}, p_n|_k$ $k \geqslant 1$, if (see [1]$_2$)

(1.3) $$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |t_n - t_{n-1}|^k < \infty .$$

In the special case when $p_n = 1$ for all values of n (resp. $k = 1$), then $|\bar{N}, p_n|_k$ summability is the same as $|C, 1|_k$ (resp. $|\bar{N}, p_n|$) summability.

The series $\sum a_n$ is said to be *bounded* $[\bar{N}, p_n]_k$ $k \geqslant 1$, if (see [1]$_3$)

(1.4) $$\sum_{v=1}^{n} p_v |s_v|^k = 0(P_n) \quad \text{as} \quad n \to \infty .$$

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If we take $k=1$ (resp. $p_n=1/n$), then $[\bar{N},\ p_n]_k$ boundedness is the same as $[\bar{N},\ p_n]$ (resp. $[R,\ \log n,\ 1]_k$) boundedness.

2 – The following theorems on $|\bar{N},\ p_n|_k$ summability factors of infinite series are known.

Theorem A. $[1]_3$. If $\sum a_n$ is bounded $[\bar{N},\ p_n]_k$ and the sequences (λ_n) and (p_n) satisfy the following conditions

$$\sum_{n=2}^{m} \frac{p_n |\lambda_n|}{P_n} = 0(1) \tag{2.1}$$

$$P_m |\Delta\lambda_m| = 0(p_m |\lambda_m|) \quad \text{as} \quad m \to \infty \tag{2.2}$$

then the series $\sum a_n \lambda_n$ is summable $|\bar{N},\ p_n|_k$, $k \geq 1$.

If we take $k=1$ in this theorem, then we get the result of Singh [2].

Theorem B. $[1]_1$. Let the sequences (λ_n) and (p_n) satisfy the conditions (2.1) and (2.2). If

$$\sum_{v=1}^{n} p_v |s_v|^k = 0(P_n \gamma_n) \quad \text{as} \quad n \to \infty \tag{2.3}$$

where (γ_n) is a positive non-decreasing sequence such that

$$P_{n+1} \gamma_n \Delta(1/\gamma_n) = 0(p_{n+1}) \quad \text{as} \quad n \to \infty\ , \tag{2.4}$$

then the series $\sum a_n \lambda_n (\gamma_n)^{-1}$ is summable $|\bar{N},\ p_n|_k$ $k \geq 1$.

If we take $\gamma_n = 1$ in Theorem B, then we get Theorem A. On the other hand if we take $k=1$ in Theorem B, then we get the result of Sinha [3].

3 – The object of this paper is to prove Theorem B under the weaker conditions. We shall prove the following theorem

Theorem. *Let the sequences* (λ_n) *and* (p_n) *satisfy the condition* (2.2) *and*

$$\sum_{n=2}^{m} \frac{p_n|\lambda_n|^k}{P_n} = 0(1) \quad as \quad m \to \infty . \tag{3.1}$$

If the condition (2.3) *satisfies, then the series* $\sum a_n \lambda_n(\gamma_n)^{-1}$ *is summable* $|\bar{N}, p_n|_k$ $k \geq 1$, *where* (γ_n) *is as in* Theorem B.

It should be noted that under the conditions of Theorem B and this theorem, (2.1) implies (3.1) but not conversly. So we are weakening hypothesis replacing (2.1) by (3.1). Since λ_n is bounded, by (2.1) and (2.2) we have

$$\sum_{n=2}^{m} \frac{p_n|\lambda_n|^k}{P_n} = \sum_{n=2}^{m} \frac{p_n|\lambda_n|\,|\lambda_n|^{k-1}}{P_n} = 0(1) \sum_{n=2}^{m} \frac{p_n|\lambda_n|}{P_n} = 0(1) . \tag{3.2}$$

Hence (2.1) implies (3.1). To show that the converse it is sufficient to take $p_n = 1$, $P_n = n$, $\lambda_n = 1/\log n$, $k = 2$.

If we take $\gamma_n = 1$ in this theorem, then we get the result of Bor [1]$_4$.

4 – We need the following lemma for the proof of our theorem.

Lemma [1]$_4$. If the sequences (λ_n) and (p_n) satisfy the conditions (2.2) and (3.1), then

$$\text{(i)} \quad \lambda_n = 0(1) \qquad \text{(ii)} \quad P_n \Delta(|\lambda_n|^k) = 0(p_n|\lambda_n|^k) \quad \text{as} \quad n \to \infty .$$

5 – Proof of the Theorem. Let T_n denote the $(\bar{N}, p_n)$ mean of the series $\sum a_n \lambda_n(\gamma_n)^{-1}$. Then, by definition, we have

$$T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v \sum_{r=0}^{v} a_r \lambda_r(\gamma_r)^{-1} = \frac{1}{P_n} \sum_{v=0}^{n} (P_n - P_{v-1})\, a_v \lambda_v(\gamma_v)^{-1} \tag{5.1}$$

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1}\, a_v \lambda_v(\gamma_v)^{-1} . \tag{5.2}$$

Using Abel's transformation, we get

$$T_n - T_{n-1} = -\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v s_v \lambda_v (\gamma_v)^{-1} + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v \Delta\lambda_v s_v (\gamma_v)^{-1}$$

$$+ \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} s_v P_v \lambda_{v+1} \Delta(1/\gamma_v) + (P_n \gamma_n)^{-1} s_n p_n \lambda_n$$

$$= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}\ , \text{ say.}$$

To prove the theorem, by Minkovski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |T_{n,r}|^k < \infty \quad \text{for} \quad r = 1,\ 2,\ 3,\ 4\ . \tag{5.3}$$

Now, applying Hölder's inequality, we have

$$\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} |T_{n,1}|^k \leq \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v |s_v|^k |\lambda_v|^k \left(\frac{1}{\gamma_v}\right)^k x \left\{\frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v\right\}^{k-1}$$

$$= 0(1) \sum_{v=1}^{m} p_v |s_v|^k |\lambda_v|^k \frac{1}{\gamma_v} \left(\frac{1}{\gamma_v}\right)^{k-1} \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}}$$

$$= 0(1) \sum_{v=1}^{m} \frac{|\lambda_v|^k}{P_v \gamma_v} p_v |s_v|^k = 0(1) \sum_{v=1}^{m-1} \Delta\left(\frac{|\lambda_v|^k}{P_v \gamma_v}\right) P_v \gamma_v + 0(1) |\lambda_m|^k$$

$$= 0(1) \sum_{v=1}^{m-1} \Delta(|\lambda_v|^k) + 0(1) \sum_{v=1}^{m-1} |\lambda_{v+1}|^k \gamma_v \Delta\left(\frac{1}{\gamma_v}\right)$$

$$+ 0(1) \sum_{v=1}^{m-1} \frac{p_{v+1} |\lambda_{v+1}|^k}{P_{v+1}} + 0(1) |\lambda_m|^k\ .$$

Since $\Delta(|\lambda_v|^k) = 0(\frac{p_v}{P_v} |\lambda_v|^k)$ and $|\lambda_m|^k = 0(1)$ (by Lemma) and since $\gamma_v \Delta(\frac{1}{\gamma_v}) = 0(\frac{p_{v+1}}{P_{v+1}})$ (by (2.4)) we have

$$\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} |T_{n,1}| = 0(1) \sum_{v=1}^{m} \frac{p_v |\lambda_v|^k}{P_v} + 0(1) \sum_{v=1}^{m-1} \frac{p_{v+1} |\lambda_{v+1}|^k}{P_{v+1}}$$

$$+ 0(1) \sum_{v=1}^{m-1} \frac{p_{v+1} |\lambda_{v+1}|^k}{P_{v+1}} + 0(1) = 0(1) \quad \text{as} \quad m \to \infty$$

by virtue of the hypothesis. Using the fact that $P_m|\Delta\lambda_m| = 0(p_m|\lambda_m|)$, as in $T_{n,1}$, we have

$$\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} |T_{n,2}|^k = 0(1) \sum_{n=2}^{m+1} \frac{p_n}{P_n(P_{n-1})^k} \left\{\sum_{v=1}^{n-1} p_v|s_v| \frac{|\lambda_v|}{\gamma_v}\right\}^k$$

$$= 0(1) \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v|s_v|^k |\lambda_v|^k \left(\frac{1}{\gamma_v}\right)^k x\left\{\frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v\right\}^{k-1}$$

$$= 0(1) \sum_{v=1}^{m} p_v|s_v|^k |\lambda_v|^k \frac{1}{P_v \gamma_v} = 0(1) \quad \text{as} \quad m \to \infty .$$

Similarly, as in $T_{n,1}$, we have

$$\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} = 0(1) \sum_{v=1}^{m} \frac{p_v|s_v|^k |\lambda_v|^k}{P_v \gamma_v} = 0(1) \quad \text{as} \quad m \to \infty .$$

Finally, we have

$$\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{k-1} |T_{n,4}|^k = \sum_{n=1}^{m} \frac{p_n|s_n|^k |\lambda_n|^k}{P_n \gamma_n} \left(\frac{1}{\gamma_n}\right)^{k-1}$$

$$= 0(1) \sum_{n=1}^{m} \frac{p_n|s_n|^k |\lambda_n|^k}{P_n \gamma_n} = 0(1), \quad \text{as} \quad m \to \infty .$$

Therefore, we get

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |T_{n,r}|^k < \infty \quad \text{for} \quad r = 1,\ 2,\ 3,\ 4 .$$

This completes the proof of the theorem.

References

[1] H. Bor: [•]$_1$ *On* $|\bar{N}, p_n|_k$ *summability factors of infinite series*, Comm. Fac. Sci. Univ. Ankara, Ser. A_1 **31** (1982), 129-133; [•]$_2$ *On* $|\bar{N}, p_n|_k$ *summability factors of infinite series*, J. Univ. Kuwait Sci. **10** (1983), 37-42; [•]$_3$ *On* $|\bar{N}, p_n|_k$ *summability factors of infinite series*, Tamkang J. Math. (1) **16** (1985), 13-20; [•]$_4$ *On absolute summability factors*, Analysis (1987) (to appear).

[2] N. Singh, *On* $|\bar{N}, p_n|$ *summability factors of infinite series*, Indian J. Math. **10** (1968), 19-24.

[3] R. Sinha, *On* $|\bar{N}, p_n|$ *summability factors of infinite series*, Comm. Fac. Sci. Univ. Ankara, Ser. A **21** (1972), 100-117.

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