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Integrability conditions for almost semiquaternion structures (**)

Introduction

The almost semiquaternion structures is a degenerate, hypercomplex structure defined by the semiquaternion algebra [2], [5]. This structure is connected with other two well known hypercomplex structures: quaternion and antiquaternion structures [1], $[8]_{1,2,3}$, ...

The Author makes a detailed study of such a structure. In a first paper $[4]_2$ the existence problem for this kind of structures and the almost semiquaternion connections were studied.

1 – Let M be a differentiable manifold, $\dim M = 4n$, $\mathcal{T}^p_q(M)$, the module of tensor fields of type (p, q), $\chi(M)$ the module of vector fields on M.

Def. 1.1. The structures $SQ = (F_1, F_2, F_3)$, $F_i \in \mathcal{T}^1_1(M)$ $(i = 1, 2, 3)$ satisfying

$$(1.1)\qquad \begin{array}{lll} F_1^2 = -I & F_2^2 = F_3^2 = 0 & \text{rank } F_2 = 2n \\ F_1 F_2 = -F_2 F_1 = F_3 & F_1 F_3 = -F_3 F_2 = -F_2 & F_2 F_3 = F_3 F_2 = 0 \end{array}$$

is called *almost semiquaternion structure*, shortly SQ-*structure*, on M.

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Let us consider structures defined by two linear complex tensor fields J_1, $J_2 \in \mathcal{T}^1_1(M)$ satisfying

$$(1.2) \qquad J_1 \cdot J_2 + J_2 \cdot J_1 = \alpha I \quad (\alpha \in R,\ I \in \mathcal{T}^1_1(M) \text{ identity}).$$

This structure is appeared first in H. Wakakuwa [7]. A study of it was made by F. Tricerri in $[6]_{1,2}$.

If $\alpha^2 < 4$ we get the almost quaternion structure, if $\alpha^2 > 4$ we have the almost antiquaternion structure. If $\alpha^2 = 4$, we obtain the almost semiquaternion structure. This is a irregular case in Tricerri's study, all his considerations referring the case $\alpha^2 \neq 4$.

We shall recall some results from $[4]_2$. By studying the almost semiquaternion G-structure associated to the SQ-structure, we proved that such a structure exists only on $4n$-dimensional manifolds. Two examples of SQ-structure are given. We considered $\mathcal{V} = \operatorname{Ker} F_2$ (the vertical distribution), and a fixed distribution $\mathcal{H}$ (horizontal distribution) complementary to $\mathcal{V}$ in $T(M)$ (i.e. $T_x(M) = \mathcal{H}_x \oplus \mathcal{V}_x\ \forall x \in M$) and invariant by F_1 and denoted by h and v the corresponding projectors. From $[4]_{1,2}$, [9], we know the existence of a tensor field $F_2^* \in \mathcal{T}^1_1(M)$, $(F_2^*)^2 = 0$, called *generalized inverse* of F_2, defined in a unique way by conditions

$$(1.3) \qquad F_2^* F_2 = h \qquad vF_2^* = 0 \qquad F_2^* h = 0\ .$$

The tensor field F_2^* depends on the choice of the distribution $\mathcal{H}$.

The structure $\text{SQ}^* = (F_1,\ F_2^*,\ F_3^* = F_1 F_2^*)$ defines on M another semiquaternion structure, called *adjoint* to SQ.

We determined the set of all linear almost SQ-connections (i.e., $\nabla F_i = 0$ $(i = 1, 2, 3)$).

$$(1.4) \qquad \nabla_X Y = \mathring{\nabla}_X Y + \frac{1}{2}\{v\mathring{\nabla}_X Y - vF_1(\mathring{\nabla}_X F_1)Y + F_2^*(\mathring{\nabla}_X F_2)Y + F_3^*(\mathring{\nabla}_X F_3)Y\}$$
$$+ \frac{1}{2}\{vQ_X Y - vF_1 Q_X F_1 Y + F_2^* Q_X F_2 Y + F_3^* Q_X F_3 Y\}$$

where $\mathring{\nabla}$ is an arbitrary fixed connection, Q is an arbitrary element of $\mathcal{T}^1_2(M)$, $Q_X Y = Q(X,\ Y)$ and $X,\ Y \in X(M)$.

The SQ-connection ∇ does not depend on the choice of the distribution $\mathcal{H}$.

The problem of linear symmetric SQ-connections (torsion free connections) is a rather difficult problem. Considering the Nijenhuis tensors associated to SQ and SQ*-structures

$$N_F(X, Y) = [FX, FY] - F[FX, Y] - F[X, FY] + F^2[X, Y] \tag{1.5}$$

where F is one of the tensor fields (complex or tangent) defining SQ or SQ*-structures, and

$$\begin{aligned} N_{F_2F_2^*}(X, Y) &= [F_2X, F_2^*Y] - F_2[F_2^*X, Y] - F_2^*[X, F_2Y] \\ &+ [F_2^*X, F_2Y] - F_2^*[F_2X, Y] - F_2[X, F_2^*Y] + [X, Y]. \end{aligned} \tag{1.6}$$

we proved that if $N_{F_2} = N_{F_2^*} = N_{F_2F_2^*} = 0$, then there exists a symmetric SQ-connection.

2 - Integrability conditions for SQ-structures

Let $\text{SQ} = (F_1, F_2, F_3)$ be the structure given by (1.1), $\mathcal{V} = \operatorname{Ker} F_2$, $\mathcal{H}$ a fixed complementary distribution in $T(M)$ to $\mathcal{V}$ and $\text{SQ}^* = (F_1, F_2^*, F_3^*)$ the adjoint structure of SQ.

Firstly, we make same remarks on the integrability of the distributions $\mathcal{V}$ and $\mathcal{H}$.

Proposition 2.1. *The vertical distribution $\mathcal{V}$ is integrable if and only if* $hN_{F_2}(F_2^*X, F_2^*Y) = 0 \quad \forall X, Y \in \chi(M)$.

Proof. From (1.5) we obtain $hN_{F_2}(X, Y) = h[F_2X, F_2Y]$, ($F_2X$ is vertical field $\forall X \in \chi(M)$). By replacing X, Y by F_2^*X, F_2^*Y we have $hN_{F_2}(F_2^*X, F_2^*Y) = h[vX, vY]$ and using Frobenius Theorem we get the proof of proposition.

Remark 2.1. If $hN_{F_2}(F_2^*X, F_2^*Y) = 0$ then all other components of N_{F_2} are vanishing (the integrability of $\mathcal{V}$ is equivalent to $N_{F_2} = 0$ and with the integrability of the tangent structure F_2 (see [9]).

Similarly is proved

Proposition 2.2. *The horizontal distribution* $\mathcal{H}$ *is integrable if and only if*

$$vN_{F_2^*}(F_2X,\ F_2Y)=0 \qquad \forall X,\ Y\in\chi(M)\ .$$

Proposition 2.3. *The distributions* $\mathcal{H}$ *and* $\mathcal{V}$ *are integrable if and only if* $N_v=0$, *where* $N_v(X,\ Y)=[vX,\ vY]-v[vX,\ Y]-v[X,\ vY]+v[X,\ Y]$.

Proof. The components of N_v are

$$hN_v(X,\ Y)=h[vX,\ vY]=hN_{F_2}(F_2^*X,\ F_2^*Y)$$

$$vN_v(vX,\ vY)=vN_v(vX,\ hY)=vN_v(hX,\ vY)=0$$

$$vN_v(hX,\ hY)=v[hX,\ hY]=vN_{F_2^*}(F_2X,\ F_2Y)\ .$$

Now we intend to find a necessary and sufficient conditions for the integrability of the almost semiquaternion G-structures (Th. 2.5, Th. 2.6). We recall that a G-structure is integrable if at every $x\in M$ there exist a local map $(U,\ \varphi)$ with local coordinates $(x^1,\ \dots,\ x^{4n})$ so that in the natural frame $(\frac{\partial}{\partial x^i},\ \dots,\ \frac{\partial}{\partial x^{4n}})$ the tensor fields F_i $(i=1,\ 2,\ 3)$ have constant expressions.

Assume that $N_{F_2}(X,\ Y)=0$, $X,\ Y\in\chi(M)$. Then the almost tangent structure defined by F_2 is integrable on M. Hence, at every $x\in M$ there exist a local map $(U,\ \varphi)$ with local coordinates $(x^1,\ \dots,\ x^{2n},\ x^{2n+1},\ \dots,\ x^{4n})$ so that in the frame

$$(2.1)\qquad (\frac{\partial}{\partial x^1},\ \dots,\ \frac{\partial}{\partial x^{2n}},\ \frac{\partial}{\partial x^{2n+1}},\ \dots,\ \frac{\partial}{\partial x^{4n}})$$

F_2 have constant expression, $F_2=\begin{pmatrix}0 & 0\\ I_{2n} & 0\end{pmatrix}$, and the fields $\frac{\partial}{\partial x^{2n+1}},\ \dots,\ \frac{\partial}{\partial x^{4n}}$ are spanning the vertical distribution $\mathcal{V}$.

We suppose that in frame (2.1) F_1 is given by

$$(2.2)\qquad F_1=\begin{pmatrix}A & B\\ C & D\end{pmatrix}$$

where $A,\ B,\ C,\ D$ are $2n$-dimensional matrices depending on $x\in M$.

The conditions $F_1F_2+F_2F_1=0$ and $F_1^2=-I$ show that

$$F_1=\begin{pmatrix} A & 0 \\ C & -A \end{pmatrix} \quad \text{where} \quad A^2=-I, \quad AC=CA\ . \tag{2.3}$$

Afterwards, we assume that the indices

$a, b, c, d, \ldots$	$i, j, k, l, \ldots$	$\alpha, \beta, \gamma, \delta, \ldots$ run from
$1, \ldots, 4n$	$1, \ldots, 2n$	$2n+1, \ldots, 4n$ respectively.

Then we have

$$F_1=\begin{pmatrix} A_i^j & 0 \\ C_\alpha^j & -A_\alpha^\beta \end{pmatrix} \quad \text{where} \quad A_i^jA_j^k=-\delta_i^k,\ A_\alpha^\beta C_\beta^k=C_\alpha^j A_j^k \tag{2.3'}$$

$$A_\alpha^\beta=A_{2n+i}^{2n+j}=A_i^j \quad \text{where} \quad \alpha=2n+i \quad \beta=2n+j\ .$$

If we suppose the Nijenhuis tensor field $N_{F_1}=0$, then in the local frame (2.1) we have

$$(N_{F_1})_{ab}^c=F_{1a}^{\ d}\frac{\partial F_{1b}^{\ c}}{\partial x^d}-F_{1b}^{\ d}\frac{\partial F_{1a}^{\ c}}{\partial x^d}+F_{1d}^{\ c}\frac{\partial F_{1a}^{\ d}}{\partial x^d}-F_{1d}^{\ c}\frac{\partial F_{1d}^{\ d}}{\partial x^a}=0\ . \tag{2.4}$$

If indices (a, b, c, d) take the particular values (i, β, γ, h), then taking into account (2.3) we obtain $A_i^h\frac{\partial A_\beta^\gamma}{\partial x^h}=0$. Multiplying this by A_k^i and summing with respect to i, it follows

$$\frac{\partial A_\beta^\gamma}{\partial x^k}=0\ . \tag{2.5}$$

Therefore, we obtain $\frac{\partial A_i^j}{\partial x^k}=0.$

If the indices (a, b, c, d) take the values (i, β, k, δ) from (2.4) we have

$$A_\beta^\delta\frac{\partial A_i^k}{\partial x^\delta}+C_\delta^k\frac{\partial A_\beta^\delta}{\partial x^i}=0\ .$$

Then taking into account (2.5) we get $A_\beta^\delta\frac{\partial A_i^k}{\partial x^\delta}=0$. By multiplying this by A_γ^β and summing with respect to β, we infer

$$\frac{\partial A_i^k}{\partial x^\delta}=0\ . \tag{2.5'}$$

Hence, from (2.5) and (2.5)′ we obtain that A_i^j (and A_α^β) are constant in frame (2.1) provided $N_{F_1}=0$.

Theorem 2.4. *If* ∇ *is a symmetric* SQ*-connection then the connection components* Γ_{ab}^c *are vanishing in frame* (2.1).

Proof. The existence of a symmetric SQ-connection ∇ implies the vanishing of the Nijenhuis tensors of SQ-structure (expressing these by means of torsion). Thus all considerations so far hold good. Moreover, ∇ being a symmetric SQ-connection, the condition $\nabla F_2=0$ can be written locally

$$(2.6)\qquad (\nabla_b F_2)_a^c=\frac{\partial F_{2a}^{\ c}}{\partial x^b}+\Gamma_{bd}^c F_{2a}^{\ d}-\Gamma_{ba}^d F_{2d}^{\ c}=0\ .$$

But, F_2 has a constant expression in frame (2.1) ($N_{F_2}=0$), so that $F_{2\alpha}^{\ i}=\delta_k^i$ (where $\alpha=2n+k$) and all other components vanish. It follows $\dfrac{\partial F_{2a}^{\ c}}{\partial x^b}=0$.

If in (2.6) the indices take the values:

(1) $d=i$ $a=\alpha$, then $\Gamma_{bi}^c F_{2\alpha}^{\ i}=0$ and from $F_{2\alpha}^{\ i}=\delta_k^i$ we have

$$(2.7)\qquad \Gamma_{bk}^c=0\ .$$

(2) $d=\alpha$ $c=i$, then $\Gamma_{ba}^\alpha F_{2\alpha}^{\ i}=0$ and from $F_{2\alpha}^{\ i}=\delta_k^i$ we get

$$(2.8)\qquad \Gamma_{ba}^{2n+i}=0 \quad \text{i.e.} \quad \Gamma_{ba}^\alpha=0\ .$$

From (2.7) and (2.8) it follows that the only components we have to consider are $\Gamma_{b\alpha}^i$.

But, $\Gamma_{b\alpha}^i=\Gamma_{\alpha b}^i$ (∇ is simmetric connection), then $b=\beta$. So we must prove that also $\Gamma_{\beta\alpha}^i$ are vanishing ($i=1,\ \dots,\ 2n;\ \alpha,\ \beta=2n+1,\ \dots,\ 4n$).

The condition $\nabla F_1=0$ can be written locally

$$(2.9)\qquad (\nabla_b F_1)_a^c=\frac{\partial F_{1a}^{\ c}}{\partial x^b}+\Gamma_{bd}^c F_{1a}^{\ d}-\Gamma_{ba}^d F_{1d}^{\ c}=0\ .$$

We take $a=\alpha$, $d=h$, in (2.9) and using (2.5) and (2.5)′ we obtain

$$(2.10)\qquad \frac{\partial c_\alpha^k}{\partial x^b}=\Gamma_{b\alpha}^h A_h^k\ .$$

But $\Gamma^h_{jk}=0$, so (2.10) becomes

$$(2.10)' \qquad \frac{\partial C^k_\alpha}{\partial x^j}=0 \qquad \frac{\partial C^k_\alpha}{\partial x^\beta}=\Gamma^h_{\beta\alpha}A^k_h .$$

Multiplying the second equation of (2.10)' by A^i_k and summing with respect k, we have

$$(2.11) \qquad \Gamma^i_{\beta\alpha}=-A^i_k\frac{\partial C^k_\alpha}{\partial x^\beta} .$$

On the other hand, $\Gamma^i_{\beta\alpha}\frac{\partial}{\partial x^i}=\nabla_{\frac{\partial}{\partial x^\beta}}\frac{\partial}{\partial x^\alpha}$, where the fields $\frac{\partial}{\partial x^\alpha}$ ($\alpha=2n+1, \ldots, 4n$) are spanning the vertical distribution. Considering the general expression (1.4) SQ-connections, we have $\nabla_X(vY)=v\nabla_X(vY)$. Hence $\nabla_X(vY)$ has only vertical components and thus $\Gamma^i_{\beta\alpha}=0$. Therefore all connections components Γ^c_{ab} are vanishing under the assumptions of the theorem.

Moreover, from (2.11) we have $\frac{\partial C^j_\alpha}{\partial x^\beta}=0$, so C^k_α has constant expression in frame (2.1). Hence we have

Theorem 2.5. *The almost semiquaternion structure* SQ$=(F_1, F_2, F_3)$ *given by* (1.1) *is integrable, if and only if there exists on* M, *a symmetric* SQ-*connection.*

Theorem 2.6. *The almost* SQ-*structure is integrable, if and only if it is locally flat.*

Proof. It is known that, whenever a structure is locally flat, then is integrable. Conversely, let us consider the curvature tensor of a symmetric SQ-connection

$$R_{c,ab}^{\ \ d}=\frac{\partial\Gamma^d_{bc}}{\partial x^a}-\frac{\partial\Gamma^d_{ac}}{\partial x^b}+\Gamma^d_{ae}\Gamma^e_{bc}-\Gamma^d_{be}\Gamma^e_{ac} .$$

From Theorem 2.4 we conclude that all Γ^c_{ab} are vanishing in frame (2.1), so that $R_{c,ab}^{\ \ d}=0$, and therefore the structure is locally flat.

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Summary

In this paper we study the integrability problem of the almost semiquaternion structures in the sense of G-structures.

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