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A priori estimates for a Navier-Stokes like system (**)

1 - Introduction

A priori estimates necessary for the classical solvability of the problem

$$v_{it} = \mu \Delta v_i - \sum_{k=1}^{n} v_k v_{ix_k} - (f(t,\ x,\ v))_{x_i} \tag{1}$$

$$v_i = 0 \quad \text{on } \partial\Omega \qquad v_i(0,\ x) = v_i^0(x) \qquad \operatorname{div} v = 0 \tag{2}$$

($i = 1, \ldots, n$; $n \leqslant 6$; $t \geqslant 0$; $x \in \Omega$-bounded smooth domain in R^n; $v = (v_1, \ldots, v_n)$) are presented. Using the N.D. Alikakos iteration technique [1] developed by the present author in [2] we show global in time estimates of v, and provided that v_i^0 are sufficiently small, also global in time $L^{2n+2}(\Omega)$ estimates of v_{it}. These estimates, in a standard way [7], $[2]_1$ ensure the existence of a uniformly Hölder continuous solution of the system (1), (2). For an arbitrary initial function v^0 the estimates are stated on a half-line $[T,\ \infty)$, with sufficiently large T, provided the corresponding solution v exists until time T. The existence of classical solutions of the original Navier-Stokes (N-S) problem is studied e.g. in [5], [6], asymptotic behaviour of solutions in [3], [5], $[10]_1$.

2 - Preliminaries

The standard notation $[10]_1$, [8] for the Sobolev and Hölder spaces is used. The norm of the vector-function v is equal to the sum of the norms of v_i.

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Subscripts different than i, j, k, m, n, h denote partial differentiation. By a $C^{1,2}_{\mathrm{loc}}$ $([0, \infty)\times\bar{\Omega})$ solution v of (1), (2) we mean its classical solution, having continuous in any compact subset of $[0, \infty)\times\bar{\Omega}$ derivatives v_{it}, v_{ix_j}, $v_{ix_jx_k}$ $(i, j, k=1, \dots, n)$. We also need the inequality

$$(3) \qquad \forall \delta>0 \quad \forall C_\delta>0 \quad \forall f\in H^1_0(\Omega) \qquad \|f\|^2_{L^2(\Omega)} \leqslant \delta\|f_x\|^2_{L^2(\Omega)} + C_\delta\|f\|^2_{L^1(\Omega)}$$

(following from the Nirenberg-Gagliardo estimates [1], [7] Chapt. II, Th. 2.2.), where $f_x=(f_{x_1}, \dots, f_{x_n})$ and $C_\delta=\mathrm{const}\,\delta^{-(n/2)}$, and the following version of the Sobolev Imbedding Theorem; let $B\subset R^n$ be a bounded domain with locally Lipschitz boundary, if $0<\mu=k-\dfrac{n}{p}-j<1$, then $W^{k,p}(B)\subset C^{j+\mu}(\bar{B})$ and

$$\exists c>0 \quad \forall \varphi\in W^{k,p}(B) \qquad \|\varphi\|_{C^{j+\mu}(\bar{B})} \leqslant c\|\varphi\|_{W^{k,p}(B)}\,.$$

The following simple observation is also used

Lemma 0. *Let $y, f{:}\,[0, \infty)\to R$ fulfil $(a, b>0)$*

$$y'(t)\leqslant -ay(t)+f(t) \qquad f(t)\leqslant b,$$

then
$$y(t)\leqslant \max\{y(0); \frac{b}{a}\}\,.$$

3 - Main result

Assuming that $\partial\Omega\in C^{2+\alpha}$ (with some $\alpha\in(0, 1)$), the functions f, f_{x_j}, f_{v_i} are Lipschitz continuous with respect to t, v_j and Hölder continuous (exponent α) with respect to x, all this uniformly in sets $R^+\times\bar{\Omega}\times[-M, M]$ $(R^+=[0, \infty))$ and also that in $R^+\times\bar{\Omega}\times R^n$

$$(4) \qquad \exists C, D>0 \quad \forall(t, x, v) \qquad |f(t, x, v)|\leqslant C+D\sum_k|v_k(t, x)|$$

we have

Theorem 1. *Let* $v_i^0 \in C^{2+\alpha}(\bar{\Omega})$ *satisfy compatibility conditions of the order* 0 *and* 1 *on* $\partial\Omega$ ([7], Chapt. IV), *also let the* $C^{1,2}$ *solution* v *corresponding to* v^0 *exist at least until time* T *given by the condition* (1)

$$(5) \qquad \exists_{\rho>0} \qquad -\mu+32dn^2\{|\Omega|^{3/16}[\int_\Omega \sum_i (v_i^0)^2(x)\,dx\,\exp(-2\mu\lambda T)]^{1/2}\cdot n y_4^{5/16}\}^{1/6} \leqslant -\rho$$

($|\cdot|$ - *denotes the Lebesgue measure, the constant* d *is defined after formula* (13), λ *in the Poincaré inequality and* y_k *are defined recursively in Lemma* 1) *then* v *is a* $C^{1,2}_{\rm loc}(R^+\times\bar{\Omega})$ *solution satisfying the additional condition*

$$(6) \qquad \|v\|_{C^{1+(\beta/2),2+\beta}(R^+\times\bar{\Omega})} \leqslant \text{const} \qquad t>T$$

with some $\beta\in(0,\ \alpha]$. *Validity of* (6) *as a priori estimate is in turn sufficient to prove the existence of a* $C^{1+(\beta/2),2+\beta}$ *solution of* (1), (2).

The proof is divided into three parts. The first part is formulated in

Lemma 1. *For arbitrary* p $(1\leqslant p<\infty)$

$$\|v(t,\ \cdot)\|_{L^p(\Omega)}\to 0 \qquad t\to\infty$$

(*conditions* $n\leqslant 6$ *and* (5) *are superfluous in Lemma* 1).

Proof. Multiplying (1) in $L^2(\Omega)$ by $v_i^{2^m-1}$ $(m=1,\ 2,\ \dots)$ and summing with respect to i, we obtain

$$(7) \qquad 2^{-m}\frac{d}{dt}\int_\Omega \sum_i v_i^{2^m}(t,\ x)\,dx$$
$$=-\mu\int_\Omega \sum_{i,k} v_{ix_k}(v_i^{2^m-1})_{x_k}\,dx-\int_\Omega \sum_{i,k} v_k v_{ix_k} v_i^{2^m-1}\,dx-\int_\Omega \sum_i f_{x_i} v_i^{2^m-1}\,dx\,.$$

The right side components are transformed in the following way

$$-\mu\int_\Omega \sum_{i,k} v_{ix_k}(v_i^{2^m-1})_{x_k}\,dx=-\mu\frac{2^m-1}{2^{2m-2}}\int_\Omega \sum_{i,k}[(v_i^{2^{m-1}})_{x_k}]^2\,dx$$

$$\int_\Omega \sum_{i,k} v_k v_{ix_k} v_i^{2^m-1}\,dx=2^{-m}\int_\Omega \sum_{i,k} v_k(v_i^{2^m})_{x_k}\,dx=-2^{-m}\int_\Omega \sum_i v_i^{2^m}(\sum_k v_{kx_k})\,dx=0\,.$$

(1) Such T is equal zero for v^0 small in the $(L^2(\Omega))^n$ norm.

For $m=1$ the last component in (7) vanishes ($\operatorname{div} v=0$) and (7) takes the simple form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}\sum_{i} v_i^2(t, x)\,\mathrm{d}x = -2\mu\int_{\Omega}\sum_{i}[(v_i)_{x_k}]^2\,\mathrm{d}x\,.$$

With the use of the Poincaré inequality

$$\exists\lambda>0 \quad \forall\varphi\in H_0^1(\Omega) \qquad \lambda\|\varphi\|^2_{L^2(\Omega)}\leqslant\sum_{i}\|\varphi_{x_i}\|^2_{L^2(\Omega)}$$

the last ensures exponential decay of v to zero in $L^2(\Omega)$

$$\int_{\Omega}\sum_{i} v_i^2(t, x)\,\mathrm{d}x\leqslant\int_{\Omega}\sum_{i}(v_i^0)^2(x)\,\mathrm{d}x\,\exp(-2\mu\lambda t)\,. \tag{8}$$

For $m=2, 3, \ldots$ the last component in (7) is estimated using the assumption (4)

$$|\int_{\Omega}\sum_{i} f_{x_i} v_i^{2^m-1}\,\mathrm{d}x| = |\int_{\Omega} f\sum_{i}(v_i^{2^m-1})_{x_i}\,\mathrm{d}x|$$

$$\leqslant\frac{2^m-1}{2^{m-1}}\int_{\Omega}(C+D\sum_{k}|v_k|)\sum_{i}|(v_i^{2^{m-1}})_{x_i}|\,|v_i^{2^{m-1}-1}|\,\mathrm{d}x = : J\,.$$

The simple estimate $|v_i^{2^{m-1}-1}|\leqslant(v_i^{2^{m-2}}+v_i^{2^{m-1}})$ and the Young inequality

$$|v_k v_i^{2^{m-1}-1}|\leqslant\frac{2^{m-1}-1}{2^{m-1}}v_i^{2^{m-1}}+\frac{1}{2^{m-1}}v_k^{2^{m-1}}$$

together with Cauchy and Hölder inequalities give

$$J\leqslant\varepsilon\int_{\Omega}\sum_{i}[(v_i^{2^{m-1}})_{x_i}]^2\,\mathrm{d}x+\mathrm{const}\,\varepsilon^{-1}[\int_{\Omega}\sum_{i} v_i^{2^m}\,\mathrm{d}x+\int_{\Omega}\sum_{i} v_i^{2^{m-1}}\,\mathrm{d}x]\,.$$

Fixing $\varepsilon(m)=\varepsilon_m$ such that $-\mu\frac{2^m-1}{2^{2m-2}}+\varepsilon_m=-\frac{\mu}{2^m}$ and hence $\varepsilon_m\sim 2^{-m}$, using the estimate (3) we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}\sum_{i} v_i^{2^m}\,\mathrm{d}x \tag{9}$$

$$\leqslant[-\frac{\mu}{\delta}+2^m\,\mathrm{const}\,\varepsilon_m^{-1}]\int_{\Omega}\sum_{i} v_i^{2^m}\,\mathrm{d}x+2^m\frac{\mu}{\delta}C_\delta\sum_{i}(\int_{\Omega} v_i^{2^{m-1}}\,\mathrm{d}x)^2+2^m\,\mathrm{const}\,\varepsilon_m^{-1}\int_{\Omega}\sum_{i} v_i^{2^{m-1}}\,\mathrm{d}x\,.$$

Next, fixing $\delta(m)=\delta_m$ $(\sim 2^{-2m})$ such that $-\dfrac{\mu}{\delta_m}+2^m \operatorname{const} \varepsilon_m^{-1}=-\mu$ and denoting by

$$y_k := \sup_{t\geq 0}\int_\Omega \sum_i v_i^{2^k}(t,\ x)\,dx \qquad (k=1,\ 2,\ \dots)$$

estimate (9) then gives

$$\frac{d}{dt}\int_\Omega \sum_i v_i^{2^m}\,dx \leq -\mu\int_\Omega \sum_i v_i^{2^m}\,dx + \operatorname{const} 2^{(3+(n/2))m} y_{m-1}^2 + \operatorname{const}' 2^{2m} y_{m-1}$$

with the constants independent of m. The last differential inequality, with the use of Lemma 0, ensures that

$$\int_\Omega \sum_i v_i^{2^m}(t,\ x)\,dx \leq \max\{\int_\Omega \sum_i (v_i^0)^{2^m}(x)\,dx;\ \operatorname{const} 2^{(3+(n/2))m}[y_{m-1}^2+y_{m-1}]\}$$

or further, for the supremum on the left side, that

$$y_m \leq \max\{\int_\Omega \sum_i (v_i^0)^{2^m}(x)\,dx;\ \operatorname{const} 2^{(3+(n/2))m}[y_{m-1}^2+y_{m-1}]\}\ . \tag{10}$$

Since in (8) we have shown that $y_1\leq \int_\Omega \sum_i (v_i^0)^2(x)\,dx$, then (10) with an easy induction argument ensures boundedness of all y_m $(m\in N)$ in the explicit form

$$\sup_{t\geq 0}\int_\Omega \sum_i v_i^{2^m}(t,\ x)\,dx \leq y_m < \infty \qquad (m=1,\ 2,\ 3,\ \dots)\ .$$

The last, using again (8) and the Hölder inequality

$$\left|\int_\Omega g_1\cdot\ldots\cdot g_k\,dx\right| \leq \|g_1\|_{L^{\alpha_1}}\cdot\ldots\cdot\|g_k\|_{L^{\alpha_k}} \qquad (\alpha_1^{-1}+\ldots+\alpha_k^{-1}=1)$$

gives for all $p\in(1,\ 2^{m-1}+1)$ $(m=1,\ 2,\ 3,\ \dots)$ the estimate

$$\begin{aligned} &\int_\Omega \sum_i v_i^p(t,\ x)\,dx \\ &\leq \sum_i \{(\int_\Omega v_i^2(t,\ x)\,dx)^{1/2}(\int_\Omega v_i^{2^m}(t,\ x)\,dx)^{(p-1)2^{-m}}|\Omega|^{p_0}\} \\ &\leq |\Omega|^{p_0}[\int_\Omega \sum_i (v_i^0)^2(x)\,dx\ \exp(-2\mu\lambda t)]^{1/2}\, n y_m^{(p-1)2^{-m}} \to 0 \end{aligned} \tag{11}$$

when $t\to\infty$ $\left(p_0=\dfrac{2^m-2(p-1)}{2^{m+1}}\right)$ and completes the proof of Lemma 1.

Remark 1. The procedure as used in $[2]_1$ (Theorem 1) or in $[2]_2$ (Lemma 2.1) applied to the recurrence estimates (10), with the use of global in time boundedness of v in $L^2(\Omega)$ (v.(8)) ensures also the estimate

$$\|v(t,\ \cdot)\|_{L^\infty(\Omega)} \leqslant \text{const} \qquad t \geqslant 0$$

where the const depends only on μ, Ω, C, D and $\|v^0\|_{L^\infty(\Omega)}$. We omit the proof here.

Remark 2. Note that since the $L^p(\Omega)$ norms of $v_k(t,\ \cdot)$ converge to 0 $(t \to \infty)$, then for sufficiently large $t \geqslant 0$ the «coefficients» v_k in the nonlinear term of (1) are arbitrarily small in $L^p(\Omega)$ and this term will be dominated by the main (Laplacian) part. This shows that the asymptotic behaviour of solutions, if they exist long enough, is generated by the diffusion term.

It is well known (e.g. [7]) that the solutions of partial differential equations will cease to exist in finite time when the solution alone, or its derivatives, become unbounded. As shown in $[2]_2$, for global in time existence of solutions u of quasilinear parabolic problems it suffices to justify global boundedness of $u(t,\ \cdot)$, $u_{x_j}(t,\ \cdot)$ and $u_t(t,\ \cdot)$ in $L^{2n+2}(\Omega)$, $L^{n+1}(\Omega)$ and $L^{2n+2}(\Omega)$, respectively. Our second step in this directions is an a priori estimate of the time derivatives v_{it}.

Lemma 2. *For every initial condition $v_i^0 \in C^{2+\alpha}(\bar{\Omega})$ satisfying compatibility conditions of the orders* 0 *and* 1 *on $\partial\Omega$ there exists time T depending only on μ, C, D, Ω and $\|v^0\|_{C^0(\bar{\Omega})}$, such that if the corresponding to v^0 $C^{1,2}_{\text{loc}}$ solution v exists for $t \in R^+$, then*

$$\|v_t(t,\ \cdot)\|_{L^{2n+2}(\Omega)} \leqslant \text{const} \qquad t \geqslant T\,.$$

Proof. The considered solution $(C^{1,2}_{\text{loc}})$ does not usually have the derivatives v_{itt}. Hence we use instead the concept of the Steklov average in our estimates, denoting the average of φ by $\varphi_h(t) := h^{-1} \int_t^{t+h} \varphi(z)\, dz$ $(h > 0)$. We need the following properties of the average

$$\varphi_{ht} = \varphi_{th} \qquad [f(t)\, g(t)]_{ht} = f(t+h)\, g_{ht}(t) + f_{ht}(t)\, g(t)\,.$$

Taking the average of both sides of (1), differentiating the result with respect to t and multiplying by $v_{iht}^{2^m-1}$ $(m=1, 2, \ldots)$, we obtain

$$2^{-m}\frac{\mathrm{d}}{\mathrm{d}t}(v_{iht}^{2^m}) = \mu(\Delta v_{iht})\, v_{iht}^{2^m-1} - (\sum_k v_k\, v_{ix_k})_{ht}\, v_{iht}^{2^m-1} - f_{x_iht}\, v_{iht}^{2^m-1} \; .$$

Summing and integrating we verify that

$$(12) \qquad 2^{-m}\frac{\mathrm{d}}{\mathrm{d}t}\sum_i \int_\Omega v_{iht}^{2^m}\,\mathrm{d}x$$

$$= -\mu\frac{2^m-1}{2^{2m-2}}\sum_i \int_\Omega \sum_k [(v_{iht}^{2^{m-1}})_{x_k}]^2\,\mathrm{d}x - \sum_i \int_\Omega \sum_k v_k(t+h,\; x)\, v_{ix_kht}\, v_{iht}^{2^m-1}\,\mathrm{d}x$$

$$- \sum_i \int_\Omega \sum_k v_{kht}\, v_{ix_k}\, v_{iht}^{2^m-1}\,\mathrm{d}x - \sum_i \int_\Omega f_{x_iht}\, v_{iht}^{2^m-1}\,\mathrm{d}x \; .$$

Note that for $m=1$ the last term in (12) vanishes $(\operatorname{div}(v_{ht})=0)$

$$(13) \qquad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_i \int_\Omega v_{iht}^2\,\mathrm{d}x = -\mu\sum_i \int_\Omega \sum_k [(v_{iht})_{x_k}]^2\,\mathrm{d}x$$

$$- \sum_i \int_\Omega \sum_k v_k(t+h,\; x)\, v_{ix_kht}\, v_{iht}\,\mathrm{d}x - \sum_i \int_\Omega \sum_k v_{kht}\, v_{ix_k}\, v_{iht}\,\mathrm{d}x \; .$$

For arbitrary m

$$\Big| - \sum_i \int_\Omega \sum_k v_k(t+h,\; x)\, v_{ix_kht}\, v_{iht}^{2^m-1}\,\mathrm{d}x\Big|$$

$$= \frac{1}{2^{m-1}}\Big|\sum_i \int_\Omega \sum_k v_k(t+h,\; x)\, (v_{iht}^{2^{m-1}})_{x_k}\, v_{iht}^{2^{m-1}}\,\mathrm{d}x\Big|$$

$$\leqslant \frac{1}{2^{m-1}}\|v(t+h,\;\cdot)\|_{L^6(\Omega)}\|v_{ht}^{2^{m-1}}\|_{L^3(\Omega)}\sum_{i,k}\|(v_{iht}^{2^{m-1}})_{x_k}\|_{L^2(\Omega)} =: K \; .$$

Since, as a consequence of Sobolev Theorem [**10**]$_1$, for $n\leqslant 6$

$$\|v_{iht}^{2^{m-1}}\|_{L^3(\Omega)} \leqslant d\sum_k \|(v_{iht}^{2^{m-1}})_{x_k}\|_{L^2(\Omega)} \qquad d=d(n,\; \Omega)$$

then

$$K \leqslant \frac{dn^2}{2^{m-1}}\|v(t+h,\;\cdot)\|_{L^6(\Omega)}\sum_{i,k}\|(v_{iht}^{2^{m-1}})_{x_k}\|_{L^2(\Omega)}^2 \; .$$

Transforming the next component

$$-\sum_i \int_\Omega \sum_k v_{kht} v_{ix_k} v_{iht}^{2^m-1} \mathrm{d}x$$

$$= -\sum_{i,k} \int_{\partial\Omega} v_{kht} v_{iht}^{2^m-1} \cos(n,\ x_k)\, \mathrm{d}\sigma + \sum_i \int_\Omega \big(\sum_k v_{kx_k}\big)_{ht} v_i v_{iht}^{2^m-1} \mathrm{d}x + \sum_{i,k} \int_\Omega v_{kht} v_i (v_{iht}^{2^m-1})_{x_k} \mathrm{d}x$$

$$= \frac{2^m-1}{2^{m-1}} \sum_{i,k} \int_\Omega v_{kht} v_i v_{iht}^{2^{m-1}-1} (v_{iht}^{2^{m-1}})_{x_k} \mathrm{d}x = : L$$

and using the Young inequality

$$|v_{kht} v_{iht}^{2^{m-1}-1}| \leqslant \frac{1}{2^{m-1}} v_{kht}^{2^{m-1}} + \frac{2^{m-1}-1}{2^{m-1}} v_{iht}^{2^{m-1}}$$

we shall estimate L in the same way as K above. For the last component in (10) we have

$$-\sum_i \int_\Omega f_{x_i ht} v_{iht}^{2^m-1} \mathrm{d}x$$

$$= -\sum_i \int_{\partial\Omega} f_{ht} v_{iht}^{2^m-1} \cos(n,\ x_i)\, \mathrm{d}\sigma + \sum_i \int_\Omega f_{ht} (v_{iht}^{2^m-1})_{x_i} \mathrm{d}x$$

$$\leqslant \sum_i \int_\Omega |f_{ht}| \frac{2^m-1}{2^{m-1}} |(v_{iht}^{2^{m-1}})_{x_i}| \cdot |v_{iht}^{2^{m-1}-1}|\, \mathrm{d}x = : N\ .$$

Then, as a consequence of the Lipschitz continuity of f and Remark 1 (the solution v is a priori bounded in $L^\infty(\Omega)$, hence the Lipschitz constants for f will be taken global in time)

$$|f_{ht}(t,\ x,\ v(t,\ x))| = \frac{1}{h} |f(t+h,\ x,\ v(t+h,\ x)) - f(t,\ x,\ v(t,\ x))|$$

$$\leqslant E + F\frac{1}{h} \sum_k |v_k(t+h,\ x) - v_k(t,\ x)| = E + F\sum_k |v_{kht}(t,\ x)|$$

(for E, F-respective Lipschitz constants for f) and the estimate of N coincides with that of J in Lemma 1. Collecting the estimates we thus have

$$2^{-m} \frac{\mathrm{d}}{\mathrm{d}t} \sum_i \int_\Omega v_{iht}^{2^m} \mathrm{d}x \tag{14}$$

$$\leqslant \Big[-\mu \frac{2^m-1}{2^{2m-2}} + \frac{dn^2}{2^{m-1}} \|v(t+h,\ \cdot)\|_{L^6(\Omega)} + dn^2 \frac{2^m-1}{2^{m-1}} \|v(t,\ \cdot)\|_{L^6(\Omega)}\Big] \sum_i \int_\Omega \sum_k [(v_{iht}^{2^{m-1}})_{x_k}]^2 \mathrm{d}x$$

$$+ \varepsilon \int_\Omega \sum_i [(v_{iht}^{2^{m-1}})_{x_i}]^2 \mathrm{d}x + \mathrm{const}\, \varepsilon^{-1} \Big[\int_\Omega \sum_i v_{iht}^{2^{m-1}} \mathrm{d}x + \int_\Omega \sum_i v_{iht}^{2^m} \mathrm{d}x\Big]\ .$$

Applying the inequality (3) to the last component in (14) we obtain

$$\frac{d}{dt}\int_\Omega \sum_i v_{iht}^{2^m}\,dx \tag{15}$$

$$\leq [-\mu\frac{2^m-1}{2^{m-2}} + 2^{m+1} dn^2 \sup_{\tau\geq t}\|v(\tau,\ \cdot)\|_{L^6(\Omega)} + 2^m\varepsilon + 2^m \operatorname{const}\varepsilon^{-1}\delta]$$

$$\cdot \sum_i \int_\Omega \sum_k [(v_{iht}^{2^{m-1}})_{x_k}]^2\,dx + 2^m \operatorname{const}\varepsilon^{-1}\{C_\delta \sum_i (\int_\Omega v_{iht}^{2^{m-1}}\,dx)^2 + \int_\Omega \sum_i v_{iht}^{2^{m-1}}\,dx\}\ .$$

For $m\in N$, $2\leq\frac{2^m-1}{2^{m-2}}\leq 4$, hence fixing $\varepsilon=\varepsilon_m$ and then $\delta=\delta_m$ both small such that

$$2^m\varepsilon_m + 2^m \operatorname{const}\varepsilon_m^{-1}\delta_m \leq \mu$$

the first bracket $[\cdot]$ in (15) is dominated by

$$[\cdot]\leq -\mu + 2^{m+1} dn^2 \sup_{\tau\geq t}\|v(\tau,\ \cdot)\|_{L^6(\Omega)}$$

and further, as a consequence of (11) with $m=4$, by

$$[\cdot]\leq -\mu + 32dn^2\{|\Omega|^{3/16}[\int_\Omega \sum_i (v_i^0)^2(x)\,dx \exp(-2\mu\lambda t)]^{1/2} n y_4^{5/16}\}^{1/6} \tag{16}$$

(note that enlarging, if necessary, the constant d, estimate (16) remains valid also for $m=2,\ 3$). Now, if the considered solution exists for all times $t\in[0,\ T]$, with sufficiently large T, such that the right side of (16) for $t=T$ is strictly negative ($\leq -\rho<0$), then (15) takes the simple form

$$\frac{d}{dt}\int_\Omega \sum_i v_{iht}^{2^m}(t,\ x)\,dx \tag{17}$$

$$\leq -\rho\sum_i \int_\Omega \sum_k [(v_{iht}^{2^{m-1}})_{x_i}]^2\,dx + 2^m \operatorname{const}\varepsilon_m^{-1}[C_{\delta_m} \sum_i (\int_\Omega v_{iht}^{2^{m-1}}\,dx)^2 + \int_\Omega \sum_i v_{iht}^{2^{m-1}}\,dx]$$

$t>T$, $1<m\leq 4$.

For $m=1$, as follows from (13), the divergence condition and the estimates following (13)

$$\frac{1}{2}\frac{d}{dt}\sum_i \int_\Omega v_{iht}^2\,dx = -\mu\sum_i \int_\Omega \sum_k [(v_{iht})_{x_k}]^2\,dx$$

$$-\sum_i \int_\Omega \sum_k v_k(t+h,\ x)\, v_{ihtx_k} v_{iht}\, dx + \sum_i \int_\Omega \sum_k v_{kht} v_i v_{ihtx_k}\, dx$$

$$\leqslant [-\mu + dn^2(\|v(t+h,\ \cdot)\|_{L^6(\Omega)} + \|v(t,\ \cdot)\|_{L^6(\Omega)})] \sum_{i,k} \|(v_{iht})_{x_k}\|^2_{L^2(\Omega)} .$$

Then, as a consequence of (11), for sufficiently large $T_1 > 0$ the bracket in the last estimate satisfies

$$[-\mu + 2dn^2 \sup_{\tau \geqslant T_1} \|v(\tau, \cdot)\|_{L^6(\Omega)}] \leqslant -\rho < 0$$

(it is easy to see that $T_1 \leqslant T$, where T was descrribed previously) and as a result of the Poincaré inequality we have

$$\frac{1}{2} \frac{d}{dt} \sum_i \int_\Omega v_{iht}^2\, dx \leqslant -\rho\lambda \sum_i \int_\Omega v_{iht}^2\, dx \qquad t \geqslant T_1$$

ensuring the exponential decay of $\|v_{ht}(t,\ \cdot)\|_{L^2(\Omega)}$ to zero $(t \to \infty)$.

An easy induction based on the recursive estimate (17) shows global in time (for $t \geqslant T$) boundedness of the $L^{2^4}(\Omega)$ norm of v_{ht}

$$\|v_{ht}(t,\ \cdot)\|_{L^{16}(\Omega)} \leqslant \text{const} \qquad t \geqslant T .$$

Passing with h to 0^+ in the above estimate and noting that in our considerations $n \leqslant 6$, we close the proof of Lemma 2.

Note, that it is easy to show the $L^p(\Omega)$ $(1 \leqslant p < \infty)$ convergence of v_t to zero $(t \to \infty)$ in the analogous way as this was done for v in the estimate (11).

Our finale step in the proof of Theorem 1 is fully analogous to the considerations in [2]$_1$ or [2]$_2$. Consider (1) with fixed $t > T$ as an elliptic system

$$(18) \qquad \mu \Delta v_i - \sum_k v_k v_{ix_k} - \sum_k f_{v_k}(t,\ x,\ v)\, v_{kx_i} = v_{it} + f_{x_i}(t,\ x,\ v) = : g_i(t,\ x) .$$

As a consequence of our assumptions concerning f and Lemmas 1, 2 the norms of the «coefficients» v_k, f_{v_k} in $L^{2n+2}(\Omega)$ are bounded uniformly for $t > 0$ and the «right side» g is bounded uniformly for $t > T$ in $(L^{2n}(\Omega))^n$. Hence using Calderon-Zygmund type estimates for linear elliptic systems (see e.g. [8] Chapt. VII, § 4) we verify that $v_i(t,\ \cdot) \in W^{2,2n}(\Omega)$ $t \geqslant T$, moreover

$$\|v_i(t,\ \cdot)\|_{W^{2,2n}(\Omega)} \leqslant \text{const} \|g_i(t,\ \cdot)\|_{L^{2n}(\Omega)} \qquad t > T .$$

Further, from the Sobolev Imbedding Theorem (since $\frac{1}{2} = 2 - \frac{n}{2n} - 1$)

$$\|v_{ix_k}(t, \cdot)\|_{C^{1/2}(\bar{\Omega})} \leq \text{const}\, \|v_i(t, \cdot)\|_{W^{2,2n}(\Omega)} . \tag{19}$$

As a direct consequence of Lemmas 1, 2 and (19), for the $C^{1,2}_{\text{loc}}$ solution v of (1), (2) v_{it}, $v_{ix_k} \in L^{2n}(D^z_\tau)$, where

$$D^z_\tau = \{(t, x) : t \in [\tau, \tau + z] \wedge x \in \bar{\Omega}\} \qquad (z > 0 \text{ fixed, } \tau > T \text{ arbitrary})$$

with the norms of v_{it}, v_{ix_k} bounded independently on $\tau > T$.

The last, due to the Sobolev Imbedding Theorem (in R^{n+1}), gives

$$v_i \in C^{\gamma,\gamma}(D^z_\tau) \qquad \gamma = 1 - \frac{n+1}{2n} \tag{20}$$

with the $C^{\gamma,\gamma}(D^z_\tau)$ norms of v_i bounded independently on $\tau > T$. Estimates (19), (20) and Lemma 3.1, Chapt. II of [7] ensure the estimate

$$v_{ix_k} \in C^{(\delta/2),\delta}(D^z_\tau) \qquad \delta = \frac{\gamma}{3} \qquad \tau > T \tag{21}$$

(the $C^{(\delta/2),\delta}$ norms are bounded independently on τ). Finally, the classical Schauder Estimates for linear parabolic equations (e.g. [7], Chapt. IV) applied to the separate equation of (1) in D^z_τ (the «coefficients» v_k, $f_{x_i}(t, x, v)$ are uniformly Hölder continuous (20)), give

$$\|v_i\|_{C^{1+(\beta/2),2+\beta}(D^z_\tau)} \leq \text{const} \qquad \beta = \min\{\alpha; \delta\} \tag{22}$$

with const independent of $\tau > T$. The arbitrariness of τ in (22) shows that (6) is satisfied and completes the proof of Theorem 1.

Using the Leray-Schauder Principle, the estimate (22) suffices (cf. [7], Chapt. V, § 6) to justify the existence of a $C^{1+(\beta/2),2+\beta}$ solution of the system (1), (2) for $t > T$.

4 – Remark 3. We sketch the proof of uniqueness of the $C^{1,2}$ solution of (1), (2). Let $v = (v_1, \ldots, v_n)$ and $w = (w_1, \ldots, w_n)$ be two different $C^{1,2}([0, \tau) \times \bar{\Omega})$ solutions corresponding to the same initial condition v^0. Subtracting the i-th equation for w from the i-th equation for v and denoting $u_i := v_i - w_i$, multiplying

the result in $L^2(\Omega)$ by u_i and summing with respect to i, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_i\int_\Omega u_i^2\,\mathrm{d}x$$

$$=-\mu\sum_i\int_\Omega\sum_k(u_{ix_k})^2\,\mathrm{d}x-\sum_i\int_\Omega\sum_k u_k w_{ix_k}u_i\,\mathrm{d}x-\sum_i\int_\Omega\sum_k v_k u_{ix_k}u_i\,\mathrm{d}x$$

$$-\sum_i\int_\Omega\big(f(t,\ x,\ v)-f(t,\ x,\ w)\big)_{x_i}u_i\,\mathrm{d}x\,.$$

Using the Cauchy inequality, Lipschitz continuity of f_{x_i} with respect to the functional argument and remembering that the $C^{1,2}$ solutions v, w are bounded for $t\in[0,\ (1-\varepsilon)\tau]$ $(\varepsilon<1)$ together with their derivatives, estimating the exemplary component as follows

$$\Big|\sum_i\int_\Omega\sum_k v_k u_{ix_k}u_i\,\mathrm{d}x\Big|\leqslant\mu\sum_i\int_\Omega\sum_k(u_{ix_k})^2\,\mathrm{d}x+\mathrm{const}\sum_i\int_\Omega u_i^2\,\mathrm{d}x$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_i\int_\Omega u_i^2\,\mathrm{d}x\leqslant\mathrm{const}'\sum_i\int_\Omega u_i^2\,\mathrm{d}x$$

and through the Gronwall inequality that

$$\sum_i\int_\Omega u_i^2(t,\ x)\,\mathrm{d}x=0\qquad\text{for}\qquad t\in[0,\ (1-\varepsilon)\tau]\,.$$

Remark 4. Let us discuss the evolution in time of the solution of (1), (2) described above. For a given smooth initial condition v^0 the existence of a local in time, uniformly Hölder continuous solution follows from the results of [7], Chapt. VII, § **6**, **7** (see also [4], [6]). When the solution exists, its $L^p(\Omega)$ norms are all the time dominated by the decreasing functions $c_p\exp(-\alpha_p t)$ $(c_p,\ \alpha_p>0)$. The significance of the nonlinear term decreases and the final picture is determined by the Laplacian part. This picture will evidently be changed under the influence of exterior forces.

Remark 5. If we wish to consider the exact form of the N-S system, we must have additional estimates of the pressure p as a functional of v. Such estimates are known in the literature (cf. [**10**]$_2$ for the inviscous case), but the full problem becames more complicated. Note that our assumption (4) is not satisfied by the original N-S system.

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Abstrakt

Es wurden gleichmässig Höldersche a priori Abschätzungen der Lösungen des Problems (1), (2) *bewiesen. Diese eignen sich gut für einen Existenzbeweis der Lösungen von* (1), (2) *in der Hölderschen Klasse. Die a priori Abschätzungen werden auch ausgenutzt zum Beweis das die Lösung* v *und ihre Ableitungen* v_{it} $(t \to \infty)$ *in* $L^p(\Omega)$ *zum Null Konvergieren. Das Problem* (1), (2) *ist sehr nahe zu den klassischen Navier-Stokes' schen Gleichungen.*

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