Riv. Mat. Univ. Parma (4) **13** (1987), 201-210

N. PARHI and S. PARHI (*)

Qualitative behaviour of solutions of forced nonlinear third order differential equations (**)

1 – In recent years, third order homogeneous differential equations (linear and nonlinear) have been the subject of investigations for many authors, viz, Hanan [3], Lazer [6], Heidel [4], Barrett [1], Jones $[5]_{1,2}$, Erbe $[2]_{1,2}$ and Philos [9] to mention a few. They obtained sufficient conditions for oscillation and nonoscillation of solutions of these equations, proved existence of oscillatory and nonoscillatory solutions and studied asymptotic behaviour of these solutions.

It seems that no work was done on oscillation theory of non-homogeneous third order differential equations until the work of the first author [7]. In this paper we consider forced nonlinear third order differential equations of the form

$$(r(t)\,y'')' + q(t)\,(y')^{\beta} + p(t)\,y^{\alpha} = f(t) \tag{1}$$

where p, q, r and f are real-valued continuous functions on $[0, \infty)$ such that $r(t) > 0$, $q(t) \leq 0$, $p(t) \leq 0$ and $f(t) \geq 0$ and each of $\alpha > 0$ and $\beta > 0$ is a ratio of odd integers. Sufficient conditions have been obtained for oscillation and nonoscillation of solutions of (1). Also we have studied asymptotic behaviour of these solutions. Results of this paper improve some of the results in [7]. The authors in their earlier works $[8]_{1,2,3}$ studied equation (1) under different sign restrictions on coefficient functions p and q.

The motivation for our work came chiefly from a recent work due to Sitter and Tefteller [10] who studied qualitative behaviour of solutions of

$$(r(t)\,y'')'' + p(t)\,y = f(t) \tag{2}$$

(*) Indirizzo degli AA.: Department of Mathematics, Berhampur University, Berhampur-760007, IND-Orissa.

(**) Ricevuto: 27-XI-1986.

where p, q and f are real-valued continuous functions on $[0, \infty)$ such that $r(t) > 0$ and $p(t)$ is ultimately positive or ultimately negative. They proved theorems concerning oscillatory and nonoscillatory behaviour of solutions of (2) with the assumption that the oscillatory and nonoscillatory behaviour of solutions of nonhomogeneous third order differential equations of the type

$$(R(t)\,y'')' + Q(t)\,y' + P(t)\,y = F(t)$$

are known, where

$$P(t) = -\frac{(r(t)\,\omega''(t))'}{\omega^2(t)} \qquad Q(t) = \frac{r(t)\,\omega''(t)}{\omega^2(t)} \qquad R(t) = \frac{r(t)}{\omega(t)}$$

$$F(t) = \frac{1}{\omega^2(t)}\,(c + \int_a^t f(s)\,\omega(s)\,ds)$$

c being a real number and $\omega(t)$ is a nonoscillatory solution of the corresponding homogeneous fourth-order differential equation

$$(r(t)\,x'')'' + p(t)\,x = 0\,.$$

We restrict our considerations to those solutions $y(t)$ of (1) which exist on the half-line $[T_y, \infty)$, $T_y \geq 0$, and are nontrivial in any neighbourhood of infinity. We may recall that a solution $y(t)$ of (1) on $[T_y, \infty)$ is said to be *nonoscillatory* if there exists a $t_1 \geq T_y$ such that $y(t) \neq 0$ for $t \geq t_1$; $y(t)$ is said to be *oscillatory* if for any $t_1 \geq T_y$ there exist t_2 and t_3 satisfying $t_1 < t_2 < t_3$ such that $y(t_2) > 0$ and $y(t_3) < 0$; it is said to be of *Z-type* if it has arbitrarily large zeros but is ultimately nonnegative or non-positive. Equation (1) is said to be *nonoscillatory* if all solutions of (1) are nonoscillatory.

2 – In this section we obtain sufficient conditions for oscillation and nonoscillation of solutions of (1).

Theorem 2.1. *If* $\alpha = \beta = 1$ *and* $q(t)$ *is once continuously differentiable such that* $p(t) - q'(t) \geq 0$, *then* (1) *is nonoscillatory.*

Proof. Let $y(t)$ be a solution of (1) on $[T_y, \infty)$, $T_y \geq 0$. If possible, let $y(t)$ be of non-negative Z-type with consecutive double zeros at a and b ($T_y \leqslant a < b$) such that $y(t) > 0$ for $t \in (a, b)$. So there exists a $c \in (a, b)$ such that $y'(c) = 0$ and $y'(t) > 0$ for $t \in (a, c)$. Now multiplying (1) through by $y'(t)$, we get

$$(3) \qquad [r(t)\,y'(t)\,y''(t)]' = r(t)\,(y''(t))^2 - q(t)\,(y'(t))^2 - p(t)\,y(t)\,y'(t) + f(t)\,y'(t)\,.$$

Integrating (3) from a to c, we obtain

$$0 = \int_a^c r(t)\,(y''(t))^2\,dt - \int_a^c q(t)\,(y'(t))^2\,dt - \int_a^c p(t)\,y(t)\,y'(t)\,dt + \int_a^c f(t)\,y'(t)\,dt > 0$$

a contradiction.

Let $y(t)$ be of non-positive Z-type with consecutive double zeros at a and b ($T_y \leqslant a < b$). So there exists a $c \in (a, b)$ such that $y'(c) = 0$ and $y'(t) > 0$ for $t \in (c, b)$. Integrating (1) from c to b, we get

$$0 \geqslant r(b)\,y''(b) - r(c)\,y''(c) \geqslant q(c)\,y(c) + \int_c^b [q'(t) - p(t)]\,y(t)\,dt > 0$$

a contradiction.

If possible, let $y(t)$ be oscillatory with consecutive zeros at a, b and a' ($T_y \leqslant a < b < a'$) such that $y'(a) \leqslant 0$, $y'(b) \geqslant 0$, $y'(a') \leqslant 0$, $y(t) < 0$ for $t \in (a, b)$ and $y(t) > 0$ for $t \in (b, a')$. So there exist $c \in (a, b)$ and $c' \in (b, a')$ such that $y'(c) = 0$, $y'(c') = 0$, $y'(t) > 0$ for $t \in (c, b)$ and $t \in (b, c')$. Suppose that $y''(b) \geqslant 0$. Integration of (3) from b to c yields

$$0 \geqslant -r(b)\,y'(b)\,y''(b) > 0$$

a contradiction. So $y''(b) < 0$. Now integrating (1) from c to b, we obtain

$$0 \geqslant r(b)\,y''(b) - r(c)\,y''(c) \geqslant q(c)\,y(c) + \int_c^b [q'(t) - p(t)]\,y(t)\,dt > 0$$

a contradiction. Hence the theorem.

Example. $\quad (\frac{1}{4}ty'')' - (t^2 + 1 + 6t^{-1})\,y' - (t^{-1} - t^{-2})\,y = 1 + 6t^{-3} \qquad t \geqslant 2\,.$

From the above theorem it follows that all solutions of the equation are nonoscillatory. In particular, $y(t) = t^{-1}$ is one such solution.

Theorem 2.2. *If* $\lim_{t\to\infty} \frac{f(t)}{p(t)} = -\infty$, *then all bounded solutions of* (1) *are nonoscillatory.*

The proof is similar to that of Theorem 2.3. in $[8]_1$ and hence is omitted.

Theorem 2.3. *Let* $\alpha \geq 1$ *and* $\beta = 1$. *Suppose that* $p(t)$, $q(t)$ *and* $f(t)$ *are once continuously differentiable functions such that* $p'(t) \geq 0$, $q'(t) \leq 0$ *and* $f'(t) \geq 0$. *If* $\lim_{t\to\infty} \frac{q'(t)}{p(t)} = +\infty$, *then all bounded solutions of* (1) *are nonoscillatory.*

Proof. Let $y(t)$ be a bounded solution of (1) on $[T_y, \infty)$, $T_y \geq 0$, such that $|y(t)| \leq K$ for $t \geq T_y$. So there exists a $t_0 \geq T_y$ such that $q'(t) - K^{\alpha-1}\, p(t) < 0$ for $t \geq t_0$..

Let $y(t)$ be of non-negative Z-type with consecutive double zeros at a and b $(t_0 \leq a < b)$. So there exists a $c \in (a, b)$ such that $y'(c) = 0$ and $y'(t) > 0$ for $t \in (a, c)$. Now integrating

$$[r(t)\, y'(t)\, y''(t)]' = r(t)\,(y''(t))^2 - q(t)\,(y'(t))^2 - p(t)\, y^{\alpha}(t)\, y'(t) + f(t)\, y'(t) \tag{4}$$

from a to c, we get

$$0 = [r(t)\, y'(t)\, y''(t)]_a^c > 0$$

a contradiction.

Let $y(t)$ be of non-positive Z-type with consecutive double zeros at a and b $(t_0 \leq a < b)$. Now integration of (4) from a to b yields

$$0 > \int_a^b f(t)\, y'(t)\, dt - \int_a^b p(t)\, y^{\alpha}(t)\, y'(t)\, dt > -\int_a^b f'(t)\, y(t)\, dt + \frac{1}{\alpha+1}\int_a^b p'(t)\, y^{\alpha+1}(t)\, dt > 0$$

a contradiction.

Let $y(t)$ be oscillatory with consecutive zeros at a, b and a' $(t_0 \leq a < b < a')$ such that $y'(a) \leq 0$, $y'(b) \geq 0$, $y'(a') \leq 0$, $y(t) < 0$ for $t \in (a, b)$ and $y(t) > 0$ for $t \in (b, a')$. So there exist $c \in (a, b)$ and $c' \in (b, a')$ such that $y'(c) = 0$, $y'(c') = 0$ and $y'(t) > 0$ on (c, b) and (b, c'). Clearly, $y''(b) \geq 0$ leads to a contradiction. So $y''(b) < 0$. Now integrating (1) from c to b yields a contradiction because

$$\begin{aligned} 0 > r(b)\, y''(b) - r(c)\, y''(c) &\geq q(c)\, y(c) + \int_c^b [q'(t) - p(t)\, y^{\alpha-1}(t)]\, y(t)\, dt \\ &\geq \int_c^b [q'(t) - K^{\alpha-1} p(t)]\, y(t)\, dt > 0\,. \end{aligned}$$

This completes the proof of the theorem.

Example. All bounded solutions of

$$(t^3 y'')' - (t^3 + 1 + \frac{3}{t} + \frac{3}{t^2} + \frac{1}{t^3}) y' - \frac{1}{t^2} y^3 = t \qquad t \geq 2$$

are nonoscillatory. In particular, $y(t) = 1 + \frac{1}{t}$ is a bounded nonoscillatory solution of the equation.

Theorem 2.4. *If $r(t) + p(t) > 0$ for large t, then any solution $y(t)$ of* (1) *which satisfies the inequality*

$$(z'')^2 + z^\alpha z' > 0 \tag{5}$$

in any interval on which it is negative is nonoscillatory.

The proof has been omitted because the arguments to prove this theorem are similar to those of Theorems 2.1 and 2.3.

Theorem 2.5. *Let $q(t)$ be bounded and once continuously differentiable such that $q'(t) \leq 0$. If*

$$\int_0^\infty p(t)\,\mathrm{d}t > -\infty \qquad \int_0^\infty f(t)\,\mathrm{d}t = \infty \qquad \int_0^\infty \frac{\mathrm{d}t}{r(t)} = \infty$$

then bounded solutions of (1) *with $\beta = 1$ are oscillatory.*

Proof. Let $y(t)$ be a bounded solution of (1) on $[T_y, \infty)$, $T_y \geq 0$, such that $|y(t)| \leq K$ for $t \geq T_y$. We claim that $y(t)$ is oscillatory. If not, $y(t) \geq 0$ or $y(t) \leq 0$ for $t \geq t_0 \geq T_y$.

Let $y(t) \geq 0$ for $t \geq t_0 \geq T_y$. Integration of (1) from t_0 to t yields

$$r(t)\,y''(t) \geq r(t_0)\,y''(t_0) + q(t_0)\,y(t_0) + K \int_{t_0}^{t} q'(s)\,\mathrm{d}s + \int_{t_0}^{t} f(s)\,\mathrm{d}s\,.$$

Since $q(t)$ bounded and $q(t) \leq 0$ imply that $\int_{t_0}^{\infty} q'(s)\,\mathrm{d}s > -\infty$, then

$$r(t)\,y''(t) \geq L \int_{t_0}^{t} f(s)\,\mathrm{d}s$$

for large t, where $0<L<1$. This in turn implies that

$$y'(t) \geqslant y'(t_0) + L \int_{t_0}^{t} \frac{1}{r(s)} \left(\int_{t_0}^{s} f(\theta)\,\mathrm{d}\theta \right) \mathrm{d}s \, .$$

Clearly,

$$\int_0^{\infty} \frac{\mathrm{d}t}{r(t)} = \infty \quad \text{implies that} \quad \lim_{t\to\infty} \int_{t_0}^{t} \frac{1}{r(s)} \left(\int_{t_0}^{s} f(\theta)\,\mathrm{d}\theta \right) \mathrm{d}s = \infty \, .$$

Thus $y'(t) \to \infty$ as $t \to \infty$, which contradicts the boundedness of $y(t)$.

Let $y(t) \leqslant 0$ for $t \geqslant t_0 \geqslant T_y$. Integrating (1) from t_0 to t, we obtain

$$r(t)\,y''(t) \geqslant r(t_0)\,y''(t_0) - q(t)\,y(t) + K^{\alpha} \int_{t_0}^{t} p(s)\,\mathrm{d}s + \int_{t_0}^{t} f(s)\,\mathrm{d}s \geqslant L' \int_{t_0}^{t} f(s)\,\mathrm{d}s$$

for large t, where $0<L'<1$. So $y(t)>0$ for large t, a contradiction. Hence the theorem.

Remark. The above theorem may be stated as follows: Suppose that the conditions of Theorem 2.5 are satisfied. Then nonoscillatory solutions of (1) with $\beta = 1$ are unbounded.

3 – In this section we study asymptotic behaviour of nonoscillatory solutions of (1).

Theorem 3.1. *If* $|q(t)| \leqslant M$ *for large* t, $\int_0^{\infty} \frac{\mathrm{d}t}{r(t)} = \infty$ *and* $\int_0^{\infty} f(t)\,\mathrm{d}t = \infty$, *then bounded nonoscillatory solutions of* (1) *with* $\beta = 1$ *are ultimately negative.*

Proof. Let $y(t)$ be a bounded nonoscillatory solution of (1) on $[T_y, \infty)$, $T_y \geqslant 0$, such that $|y(t)| \leqslant K$ for $t \geqslant T_y$.

If possible, let $y(t) > 0$ for $t \geqslant t_0 \geqslant T_y$. Let $y'(t)$ be oscillatory (non-negative Z-type) with consecutive zeros (double zeros) at a and b $(t_0 \leqslant a < b)$ such that $y'(t) > 0$ for $t \in (a, b)$. Integration of (1) from a to b yields

$$0 \geqslant \int_a^b [f(t) - p(t)\,y^{\alpha}(t) - q(t)\,y'(t)]\,\mathrm{d}t > 0$$

a contradiction. Let $y'(t) > 0$ for large t. Now integrating (1) from t_0 to t, we obtain

$$r(t)\,y''(t) \geqslant r(t_0)\,y''(t_0) + \int_{t_0}^{t} f(s)\,ds > 0$$

for large t. This in turn implies that $y(t)$ is unbounded, a contradiction. So $y'(t) \leqslant 0$ for large t. In this case also we obtain a contradiction because integration of (1) from t_0 to t yields

$$r(t)\,y''(t) \geqslant r(t_0)\,y''(t_0) - My(t_0) + \int_{t_0}^{t} f(s)\,ds > L$$

for large t, where $L > 0$ is a constant, that is, $y'(t) > 0$ for large t.

This completes the proof of the theorem.

Example. $(\frac{1}{4}ty'')' - \frac{1}{t}y' - t^4 y^3 = t \qquad t > 1.$

Theorem 3.2. *If*

$$\lim_{t\to\infty} \frac{f(t)}{p(t)} = -\infty \qquad \int_0^{\infty} f(t)\,dt = \infty$$

then, for every bounded solution $y(t)$ *of* (1), $\lim_{t\to\infty} y(t)$ *exists.*

Proof. Let $y(t)$ be a bounded solution of (1) on $[T_y,\ \infty)$, $T_y \geqslant 0$, such that $|y(t)| \leqslant K$ for $t \geqslant T_y$. From the given condition it follows that there exists a $t_0 \geqslant T_y$ such that $f(t) + K^{\alpha} p(t) > 0$ for $t \geqslant t_0$. In view of Theorem 2.2., $y(t)$ is nonoscillatory. So it is ultimately positive or ultimately negative.

Let $y(t) > 0$ for $t \geqslant t_1 \geqslant t_0$. Proceeding as in Theorem 3.1., we can show that $y'(t)$ cannot be oscillatory or non-negative Z-type or positive for large t. So $y'(t) \leqslant 0$ for large t and hence $\lim_{t\to\infty} y(t)$ exists.

Let $y(t) < 0$ for $t \geqslant t_1 \geqslant t_0$. If $y'(t)$ is oscillatory (non-negative Z-type) with consecutive zeros (double zeros) at a and b $(t_1 \leqslant a < b)$ such that $y'(t) > 0$ for $t \in (a,\ b)$, then integrating (1) from a to b we get

$$0 \geqslant \int_a^b f(t)\,dt - \int_a^b p(t)\,y^{\alpha}(t)\,dt \geqslant \int_a^b [f(t) + K^{\alpha} p(t)]\,dt > 0$$

a contradiction. So $y'(t) > 0$ or $\leqslant 0$ for large t. In any case $\lim_{t\to\infty} y(t)$ exists. Hence the theorem.

Theorem 3.3. *If* $|q(t)|\leqslant M$ *for large* t

$$\int_0^\infty \frac{dt}{r(t)}=\infty \qquad \int_0^\infty f(t)\,dt=\infty \qquad \int_0^\infty p(t)\,dt>-\infty \qquad \lim_{t\to\infty}\frac{f(t)}{p(t)}=-\infty$$

then all solutions of (1) *with* $\beta=1$ *are unbounded.*

Proof. Suppose the contrary. Let $y(t)$ be a solution of (1) such that $|y(t)|\leqslant K$ for $t\geqslant T_y\geqslant 0$. From Theorems 2.2. and 3.1. it follows that $y(t)$ is ultimately negative. Let $y(t)<0$ for $t\geqslant t_0\geqslant T_y$. There exists a $t_1\geqslant t_0$ such that $f(t)+K^\alpha p(t)>0$ for $t\geqslant t_1$. Proceeding as in Theorem 3.2. we can show that $y'(t)$ cannot be oscillatory or non-negative Z-type. If $y'(t)>0$ for large t, then integrating (1) from t_0 to t we obtain

$$r(t)\,y''(t)\geqslant r(t_0)\,y''(t_0)+K^\alpha\int_{t_0}^t p(s)\,ds+\int_{t_0}^t f(s)\,ds\ .$$

This in turn implies that $y(t)>0$ for large t, a contradiction. So $y'(t)\leqslant 0$ for large t. Since

$$r(t)\,y''(t)\geqslant r(t_0)\,y''(t_0)-MK+K^\alpha\int_{t_0}^t p(s)\,ds+\int_{t_0}^t f(s)\,ds$$

implies that $r(t)\,y''(t)>L$ for large t, $L>0$, then $y'(t)>0$ for large t, a contradiction. Hence $y(t)$ must be unbounded and this proves the theorem.

Example. All solutions of

$$(ty'')'-\left(1-\frac{1}{t}\right)y'-\frac{1}{t^4}y^3=t^2+2t-4 \qquad t\geqslant 2$$

are unbounded. In particular, $y(t)=-t^2$ is an unbounded solution of the equation.

Theorem 3.4. *Let*

$$\int_0^\infty \frac{dt}{r(t)}=\infty \qquad \int_0^\infty f(t)\,dt=\infty \qquad \int_0^\infty p(t)\,dt>-\infty\ .$$

If $q(t)$ *is once continuously differentiable such that* $q'(t)\geqslant 0$ *and* $\int_0^\infty q'(t)\,dt<\infty$, *then all nonoscillatory solutions of* (1) *with* $\beta=1$ *are unbounded.*

The proof is similar to that of Theorem 3.3. and hence is omitted.

Example. $(ty'')' - \frac{1}{t}y' - \frac{1}{t^4}y^3 = t^2 \qquad t > 1.$

Theorem 3.5. *Let*

$$\int_0^\infty \frac{dt}{r(t)} = \infty \qquad \int_0^\infty p(t)\,dt = -\infty \qquad \int_0^\infty f(t)\,dt < \infty$$

and $q(t)$ *be bounded. If* $r(t) + p(t) > 0$ *for large* t, *then for all bounded solutions* $y(t)$ *of* (1) *with* $\beta = 1$, *which satisfy the inequality* (5) *in any interval on which it is negative,* $\lim_{t\to\infty} y(t) = 0$.

Proof. Let $|y(t)| \leqslant K$ and $|q(t)| \leqslant M$ for $t \geqslant t_0$. From Theorem 2.4 it follows that $y(t)$ is ultimately positive or ultimately negative. We shall prove the theorem for the case $y(t) > 0$ for $t \geqslant t_1 \geqslant t_0$. The proof is similar for the other case.

It is easy to show that $y'(t)$ cannot be oscillatory or non-negative Z-type. If possible, let $y'(t) > 0$. So

$$r(t)\,y''(t) \geqslant r(t_1)\,y''(t_1) - y^\alpha(t_1) \int_{t_1}^{t} p(s)\,ds$$

and hence $y(t)$ is unbounded, a contradiction. Consequently, $y'(t) \leqslant 0$ and $\lim_{t\to\infty} y(t)$ exists. If $\lim_{t\to\infty} y(t) = A > 0$, then integrating (1) from t_1 to t, we obtain

$$r(t)\,y''(t) \geqslant r(t_1)\,y''(t_1) - My(t_1) - y^\alpha(t) \int_{t_1}^{t} p(s)\,ds\ .$$

This in turn implies that $y'(t) > 0$ for large t. This contradiction proves that $\lim_{t\to\infty} y(t) = 0$. Hence the theorem.

Example. $(2ty'')' - \frac{1}{t^2}y' - ty^5 = \frac{8}{t^3} \qquad t \geqslant 1.$

$y(t) = -\frac{1}{t}$ is a solution of the equation such that $\lim_{t\to\infty} y(t) = 0$.

References

[1] J. H. BARRETT, *Oscillation theory of ordinary linear differential equations*, Adv. in Math. 3 (1969), 415-509. (Reprinted in «Lectures in Ordinary Differential Equations», Ed. R. McKelvey, Academic Press, New York, 1970). MR 41 # 2113.

[2] L. ERBE: [•]$_1$ *Oscillation, nonoscillation and asymptotic behaviour for third order nonlinear differential equations*, Ann. Mat. Pura Appl. **110** (1976), 373-393; [•]$_2$ *Existence of oscillatory solutions and asymptotic behaviour for a class of third order linear differential equations*, Pacific J. Math. 64 (1976), 369-385.

[3] M. HANAN, *Oscillation criteria for third-order linear differential equations*, Pacific J. Math. **11** (1961), 919-944. MR 26 # 2695.

[4] J. W. HEIDEL, *Qualitative behaviour of solutions of a third order nonlinear differential equation*, Pacific J. Math. **27** (1968), 507-526.

[5] G. D. JONES: [•]$_1$ *Properties of solutions of a class of third order differential equations*, J. Math. Anal. Appl. 48 (1974), 165-169; [•]$_2$ *Oscillation criteria for third order differential equations*, SIAM J. Math. Anal. 7 (1976), 13-15.

[6] A. C. LAZER, *The behaviour of solutions of the differential equation* $y''' + p(x)y' + q(x)y = 0$, Pacific J. Math. **17** (1966), 435-466. MR 33 # 1552.

[7] N. PARHI, *Nonoscillatory behaviour of solutions of nonhomogeneous third order differential equations*, Applicable Anal. **12** (1981), 273-285.

[8] N. PARHI and S. PARHI: [•]$_1$ *Oscillation and nonoscillation theorems for non-homogeneous third order differential equations*, Bull. Inst. Math. Acad. Sinica **11** (1983), 125-139; [•]$_2$ *On the behaviour of solutions of the differential equation* $(r(t)y'')' + q(t)(y')^\beta + p(t)y^\alpha = f(t)$, Annales Polon. Math. **47** (1986), 137-148; [•]$_3$ *Nonoscillation and asymptotic behaviour for forced nonlinear third order differential equations*, Bull. Inst. Math., Acad. Sinica **13** (1985), 367-384.

[9] CH. G. PHILOS, *Oscillation and asymptotic behaviour of third order linear differential equations*, Bull. Inst. Math. Acad. Sinica **11** (1983), 141-160.

[10] R. E. SITTER and S. C. TEFTELLER, *Oscillation of a fourth order nonhomogeneous differential equation*, J. Math. Anal. Appl. **59** (1977), 93-104.

Abstract

In this paper sufficient conditions have been obtained for oscillation and nonoscillation of solutions of forced nonlinear third order differential equations of the form

$$(r(t)y'')' + q(t)(y')^\beta + p(t)y^\alpha = f(t)$$

where p, q, r *and* f *are real-valued continuous functions on* $[0, \infty)$ *such that* $r(t) > 0$, $p(t) \leq 0$, $q(t) \leq 0$ *and* $f(t) \geq 0$ *and each of* $\alpha > 0$ *and* $\beta > 0$ *is a ratio of odd integers. Asymptotic behaviour of nonoscillatory solutions of these equations has also been studied.*

***