Riv. Mat. Univ. Parma (4) **13** (1987), 281-299

WILLIAM D. L. APPLING (*)

A transformation of variables characterization theorem for Stieltjes integrals (**)

1 - Introduction

In recent papers, various authors [1], [4], [5] have carried on the long standing and continuing investigation of the interplay between the differentiability, absolute continuity, continuity, bounded variation and integrability properties of real-valued functions defined on certain «standard» subsets of the real numbers.

In this paper we investigate, for the setting and notions given above, a question involving transformation of variables for Stieltjes integrals. In «standard» Stieltjes integral (see [2], [3] and 2) transformation of variables theorems, i.e., theorems having conclusions of the form

$$\int_r^s f(v(t))\,dg(v(t)) = \int_{v(r)}^{v(s)} f(x)\,dg(x)$$

subject to various conditions on f, g and v, one of the primary questions that arise is that of the existence of the integral on the left, given that the integral on the right exists. An example of such a theorem is the following, which is well known and routinely shown.

Theorem 1.1. *Suppose that* $p<q$, $a<b$, v *is a continuous monotonic function from* $[p\,;q]$ *into* $[a\,;b]$, *and each of* f *and* g *is a function from* $[a\,;b]$ *into*

(*) Indirizzo: Department of Mathematics, P.O. Box 5116, North Texas State University, Denton, USA-Texas 76203.

(**) Ricevuto: 17-II-1987.

R such that the integral (see 2) $\int_a^b f(x)\,dg(x)$ *exists. Then the integral* $\int_p^q f(v(t))\,dg(v(t))$ *exists and is* $\int_{v(p)}^{v(q)} f(x)\,dg(x)$.

A stronger version of Theorem 1.1., which we shall be using in this paper, is the following corollary, for which we give an indication of proof.

Corollary 1.1. *Suppose that $r<s$, $a<b$, D is a subdivision* (see 2) *of $[r;s]$, v is a continuous function from $[r;s]$ into $[a;b]$ such that if $[p;q]$ is in D, then v is monotonic on $[p;q]$, and each of f and g is a function from $[a;b]$ into R such that the integral $\int_a^b f(x)\,dg(x)$ exists. Then the integral $\int_r^s f(v(t))\,dg(v(t))$ exists and is $\int_{v(r)}^{v(s)} f(x)\,dg(x)$.*

Indication of proof. By Theorem 1.1., for each $[p;q]$ in D, $\int_p^q f(v(t))\,dg(v(t))$ exists and is $\int_{v(p)}^{v(q)} f(x)\,dg(x)$. We therefore have the following existence and equality

$$\int_{v(r)}^{v(s)} f(x)\,dg(x) = \sum_D \int_{v(p)}^{v(q)} f(x)\,dg(x) = \sum_D \int_p^q f(v(t))\,dg(v(t)) = \int_r^s f(v(t))\,dg(v(t)) .$$

Notice that in the above corollary, because of the monotonicity conditions on v, the only requirement on f and g is that the integral $\int_a^b f(x)\,dg(x)$ exist. We ask the following question: what kind of transformation of variables theorem can we have if we replace the monotonicity conditions of the corollary on v with bounded variation? As we shall see, even as restrictive a condition as f continuous and g of bounded variation is not enough to ensure that for v as immediately described above, the integral $\int_r^s f(v(t))\,dg(v(t))$ exists. Indeed, we shall see from Theorem 3.1., stated below, that if g is such that for every f continuous on $[a;b]$ and v, once again, as described above, the integral $\int_r^s f(v(t))\,dg(v(t))$ exists, then g must be Lipschitz on $[a;b]$.

Before stating Theorem 3.1. we give some definitions.

If $i<j$ and $S \subseteq \mathbb{R}$, then:

(i) $C[i;j]S$ is the set of all functions from $[i;j]$ into S continuous on $[i;j]$.

(ii) $BV[i\,;j]S$ is the set of all functions from $[i\,;j]$ into S having bounded variation on $[i\,;j]$.

(iii) $CBV[i;\,j]S$ is $C[i\,;j]S \cap BV[i\,;j]S$.

(iv) $AC[i\,;j]S$ is the set of all functions from $[i\,;j]$ into S, absolutely continuous on $[i\,;j]$.

We note that $AC[i\,;j]S \subseteq CBV[i\,;j]S$.

Theorem 3.1. *Suppose that $a < b$, $r < s$ and g is a function from $[a\,;b]$ into $\boldsymbol{R}$. Then the following five statements are equivalent:*

(1) *If f is in $C[a\,;b]\boldsymbol{R}$, and v is in $CBV[r\,;s][a\,;b]$, then $\int_r^s f(v(t))\,\mathrm{d}g(v(t))$ exists.*

(2) *If f is in $C[a\,;b]\boldsymbol{R}$ and v is in $AC[r\,;s][a\,;b]$, then $\int_r^s f(v(t))\,\mathrm{d}g(v(t))$ exists.*

(3) *g is in $C[a\,;b]\boldsymbol{R}$ and, if v is in $AC[r\,;s][a\,;b]$, then $g(v)$ is in $BV[r\,;s]\boldsymbol{R}$ (and therefore trivially in $CBV[r\,;s]\boldsymbol{R}$).*

(4) *There is $K > 0$ and $d > 0$ such that if $\{I_k\}_{k=1}^m$ is a finite sequence of subinterval of $[a\,;b]$ such that $\sum_{k=1}^m \Delta_{I_k} x \le d$, then $\sum_{k=1}^m |\Delta_{I_k} g| \le K$.*

(5) *g is Lipschitz on $[a\,;b]$.*

Furthermore, if g satisfies such conditions, then for each f and v as given in (1)

$$\int_r^s f(v(t))\,\mathrm{d}g(v(t)) = \int_{v(r)}^{v(s)} f(x)\,\mathrm{d}g(x)\,.$$

We pause here to note a small fact. Theorem 3.1 could have asserted the equivalence of six statements, the additional statement placed between statements (2) and (3) above and given as follows:

(2.5) *If f is in $C[a\,;b]\boldsymbol{R}$ and v is in $AC[r\,;s][a\,;b]$, then for some subdivision D of $[r\,;s]$, $\{\sum_E f(v(z(I)))\,\Delta_I g(v)$: E a refinement of D, z an interpolating function on $E\}$ is bounded.*

Clearly (2.5) follows from (2) and only the most minor modifications in the argument showing that (2) implies (3) are required to show that (2.5) implies (3). However, in the interests of orthodoxy we refrain from this insertion.

In the final section of this paper we extend the final remark of Theorem 3.1. and prove, with some labor, the following theorem.

Theorem 4.6. *Suppose that* $r<s$, $a<b$, v *is in* $CBV[r;s][a;b]$, *each of* u *and* g *is a function from* $[a;b]$ *into* $\boldsymbol{R}$, g *is Lipschitz on* $[a;b]$ *and* u *is quasi-continuous on* $[a;b]$. *Then* $\int_r^s u(v(t))\,dg(v(t))$ *exists and is* $\int_{v(r)}^{v(s)} u(x)\,dg(x)$.

2 - Preliminary definitions, lemmas and theorems

If $p<q$, then the statement that D is a subdivision of $[p;q]$ means that D is a finite collection of nonoverlapping (number) intervals whose union is $[p;q]$.

If $p<q$ and D is a subdivision of $[p;q]$, then the statement that E is a refinement of D means that E is a subdivision of $[p;q]$ such that each element of E is a subset of some element of D. We shall let «$P \ll Q$» mean that P is a refinement of Q.

Throughout this paper all integrals will be refinement-wise limits of the appropriate sums. We shall use standard notation and shall assume and use basic properties and conventions.

The remainder of this section consists of 6 lemmas and a theorem that we shall use to prove Theorem 3.1. Some of these are sufficiently self evident to warrant omitting proof.

Lemma 2.1. *Suppose that* $a<b$, p *is in* $[a;b]$, $\{w_k\}_{k=1}^{\infty}$ *is a sequence of numbers of* $[a;b]$ *such that* $\sum_{k=1}^{\infty} |w_k - p| < \infty$, $r<s$, q *is in* $[r;s]$ *and* $\{x_j\}_{j=1}^{\infty}$ *is a monotonic sequence of distinct elements of* $[r;s]$ *such that* $x_n \to q$ *as* $n \to \infty$. *Then there is a function* v *in* $AC[r;s][a;b]$ *such that for each positive integer* n, $v(x_n) = w_n$ *if* n *is even, and* $v(x_n) = p$ *if* n *is odd.*

Lemma 2.2. *Suppose that* $a<b$, p *is in* $[a;b]$, $\{z_k\}_{k=1}^{\infty}$ *is a sequence of distinct numbers of* $[a;b]$, *all distinct from* p, *such that* $z_n \to p$ *as* $n \to \infty$, *and* $\{y_k\}_{k=1}^{\infty}$ *is a sequence of numbers which converges. Then there is an element* f *of* $C[a;b]\boldsymbol{R}$ *such that if* n *is a positive integer, then* $f(z_n) = y_n$.

Lemma 2.3. *Suppose that* $l<m$, $a<b$, $0<d_1$, $0<d_2$, S *is a finite number set,* u *is an element of* $AC[l;m][a;b]$, x *is in* $[a;b]$ *and* $|u(m)-x|<d_2$. *Then there is an element* u^* *of* $AC[l;m][a;b]$ *having some element in common with* u, *such that* $u^*(m)=x$, $u^*(l)$ *is not in* S, $|u(l)-u^*(l)|<d_1$ *and* $\int_{[l;m]} |du - du^*| < d_2$.

Lemma 2.4. *Suppose that* $r<s$, $r\leq k\leq s$, $a<b$, $\{l_i\}_{i=1}^{\infty}$ *is a monotonic sequence of distinct numbers of* $[r;s]$ *such that* $l_n\to k$ *as* $n\to\infty$, *and such that, for each positive integer* n, u_n *is in* $AC[l_{n+1};\ l_n][a;b]$ *or* $AC[l_n;l_{n+1}][a;b]$, *as the case may be. Suppose further that if* n *is a positive integer, then* $u_{n+1}(l_{n+1})=u_n(l_{n+1})$, *and* $\sum_{n=1}^{\infty}\int_{[l_{n+1};l_n]}|du_n|<\infty$ *or* $\sum_{n=1}^{\infty}\int_{[l_n;l_{n+1}]}|du_n|<\infty$, *as the case may be. Then there is an element* u *of* $AC[r;s][a;b]$ *such that* $\bigcup_{n=1}^{\infty}u_n\subseteq u$.

Lemma 2.5. *Suppose that* $r<s$, v *is in* $CBV[r;s]\boldsymbol{R}$, D *is a subdivision of* $[r;s]$, S *is a finite subset of* $v([r;s])$ *and* $0<c$. *Then there is a refinement* E *of* D, *a subset* E' *of* E *and a reversible function* X *from* E' *onto* S *such that if* $[p;q]$ *is in* E', *then*

$$|X[p;q]-v(p)|+|X[p;q]-v(q)|+\int_{[p;q]}|dv|<c\,.$$

Proof. There is a refinement E of D such that if $[p;q]$ is in E, then $\int_{[p;q]}|dv|<\min\{c/3,\ \min\{|x-y|:x,\ y \text{ in } S,\ x\neq y\}\}$. Suppose that z is in S. There is w in $[r;s]$ such that $z=v(w)$. There is $[p;q]$ in E such that w is in $[p;q]$. If w' is in $[p;q]$ and $v(w')$ is in S, then

$$|v(w)-v(w')|\leq\int_{[p;q]}|dv|<\min\{|x-y|:x,\ y\ \text{in}\ S,\ x\neq y\}$$

so that $0=|v(w)-v(w')|$. Thus there is a function Y from S into E such that if z is in S, then z is the only element of S in $v(Y(z))$. Y is clearly reversible. Let $X=Y$'s inverse and $E'=Y$'s range. X is a reversible function from E' onto S. Now, if $[p;q]$ is in E', then for some w in $[p;q]$, $X[p;q]=v(w)$, so that

$$|X[p;q]-v(p)|+|X[p;q]-v(q)|+\int_{[p;q]}|dv|$$

$$=|v(w)-v(p)|+|v(w)-v(q)|+\int_{[p;q]}|dv|\leq 3\int_{[p;q]}|dv|<c\,.$$

Lemma 2.6. *Suppose that* $r<s$, v *is in* $CBV[r;s]\boldsymbol{R}$, $v([r;s])$ *contains two elements,* D *is a subdivision of* $[r;s]$, $\{x_k\}_{k=1}^{n}$ *is a finite sequence of elements of* $v([r;s])$ *and* $0<c$. *Then there is a refinement* E *of* D *and a reversible function* X

from a subset E' of E onto $\{1, \ldots, n\}$ such that if $[p; q]$ is in E', then

$$|x_{X[p;q]} - v(p)| + |x_{X[p;q]} - v(q)| + \int_{[p;q]} |\mathrm{d}v| < c .$$

Proof. Clearly, there is a reversible function W from $\{1, \ldots, n\}$ into $v([r; s])$ such that if k is in $\{1, \ldots, n\}$, then $|W(k) - x_k| < c/6$. From Lemma 2.5. it follows that there is a refinement E of D and a reversible function X from a subset E' of E onto $\{1, \ldots, n\}$ such that if $[p; q]$ is in E', then $|W(X[p; q]) - v(p)| + |W(X([p; q]) - v(q)| + \int_{[p;q]} |\mathrm{d}v| < c/3$ so that

$$|x_{X[p;q]} - v(p)| + |x_{X[p;q]} - v(q)| + \int_{[p;q]} |\mathrm{d}v| \leq |x_{X[p;q]} - W(X[p; q])| + |W(X[p; q]) - v(p)|$$

$$+ |x_{X[p;q]} - W(X[p; q])| + |W(X[p; q]) - v(q)| + \int_{[p;q]} |\mathrm{d}v| < c/6 + c/6 + c/3 < c .$$

Theorem 2.1. *Suppose that $r < s$, v is in $AC[r; s]\boldsymbol{R}$, $v([r; s])$ contains two elements, D is a subdivision of $[r; s]$, m is a positive integer, $\{[x_k; y_k]\}_{k=1}^{m}$ is a sequence of subintervals of $v([r; s])$ and $0 < c$. Then there is a refinement E of D, a reversible function X from a subset E' of E onto $\{1, \ldots, m\}$ and sequences $\{u_k\}_{k=1}^{m}$ and $\{t_k\}_{k=1}^{m}$ such that:*

(i) *if $[p; q]$ is in E', then $|x_{X[p;q]} - v(p)| + |x_{X[p;q]} - v(q)| + \int_{[p;q]} |\mathrm{d}v| < c$;*

(ii) *if $[p; q]$ is in E', then $u_{X[p;q]}$ is in $AC[p; q]v([r; s])$, $u_{X[p;q]}(p) = v(p)$, $u_{X[p;q]}(q) = v(q)$, $p < t_{X[p;q]} < q$, $u_{X[p;q]}(t_{X[p;q]}) = y_{X[p;q]}$, $\int_{[p;q]} |\mathrm{d}u_{X[p;q]}| = |y_{X[p;q]} - v(p)| + |y_{X[p;q]} - v(q)|$;*

(iii) *if, for each $[p; q]$ in E', $v_{[p;q]}$ is the restriction of v to $[p; q]$ and $\alpha = [v - U_{E'}\ v_{[p;q]}] \cup [U_{E'}\ u_{x[p;q]}]$, then α is in $AC[r; s]v([r; s])$, $\alpha(r) = v(r)$, $\alpha(s) = v(s)$, if x is in $[p; q]$ in $E - E'$, then $\alpha(x) = v(x)$ and*

$$\int_{[r;s]} |d\alpha - dv| = \sum_{E'} \int_{[p;q]} |du_{X[p;q]} - dv_{[p;q]}| \leq \sum_{E'} [|y_{X[p;q]} - v(p)| + |y_{X[p;q]} - v(q)| + \int_{[p;q]} |dv_{[p;q]}|]$$

$$\leq \sum_{E'} [|y_{X[p;q]} - x_{X[p;q]}| + |x_{X[p;q]} - v(p)| + |y_{X[p;q]} - x_{X[p;q]}| + |x_{X[p;q]} - v(q)| + \int_{[p;q]} |dv_{[p;q]}|]$$

$$< \sum_{E'} [2|y_{x[p;q]} - x_{X[p;q]}| + c] .$$

Proof. By Lemma 2.6 there is a refinement E of D and a reversible function X from a subset E' of E onto $\{1, \ldots, m\}$ such that if $[p;q]$ is in E', then $|x_{X[p;q]} - v(p)| + |x_{X[p;q]} - v(q)| + \int_{[p;q]} |dv| < c$. Thus (i) is satisfied.

For each $[p;q]$ in E', there is $t_{X[p;q]}$ such that $p < t_{X[p;q]} < q$ and an element $u_{X[p;q]}$ of $AC[p;q]v([r;s])$ such that if $p \leq x \leq q$, then $u_{X[p;q]}(x) = v(p) + [(y_{X[p;q]} - v(p))/(t_{X[p;q]} - p)]\ \max\{\min\{x-p,\ t_{X[p;q]} - p\},\ 0\} + [(v(q) - y_{X[p;q]})/(q - t_{X[p;q]})]\ \max\{\min\{x - t_{X[p;q]},\ q - t_{X[p;q]}\},\ 0\}$. Routine considerations imply that (ii) is satisfied.

Finally, (iii) is an obvious and self explanatory consequence of conditions (i) and (ii) and the definition of $u_{X[p;q]}$ for each $[p;q]$ in E'.

We end this section by stating a notational convention: If, in a given discussion, an expression is to be repeated and is of sufficient complexity, it will be enclosed in square brackets with a subscript attached; thereafter only the brackets with subscript need be written.

3 - A transformation of variables theorem

In this section we prove Theorem 3.1, as stated in the introduction.

Proof of Theorem 3.1. Obviously (1) implies (2).

We now show that (2) implies (3). Suppose that (2) is true. First, $\int_r^s (a + [(b-a)/(s-r)](t-r))\,dg(a + [(b-a)/(s-r)](t-r))$ exists, which implies that $\int_r^s t\,dg(a + [(b-a)/(s-r)](t-r))$ exists, so that, from routine considerations, $g(a + [(b-a)/(s-r)](t-r))$ is bounded on $[r;s]$. Therefore, clearly, g is bounded on $[a;b]$. Let $M = \sup\{|g(x) - g(y)| : \{x,\ y\} \subseteq [a;b]\}$.

We now show that g is continuous on $[a;b]$. Suppose, on the contrary, that there is p in $[a;b]$ such that g is not continuous at p. Then there is $c > 0$ and a sequence $\{w_k\}_{k=1}^{\infty}$ of distinct numbers of $[a;b]$, all distinct from p such that for each positive integer k, $0 < |w_k - p| < 2^{-k}$ and $|g(w_k) - g(p)| \geq c$.

Therefore $\sum_{k=1}^{\infty} |w_k - p| < \infty$. By Lemma 2.1 there is an element v of $AC[r;s][a;b]$ such that for each positive integer n, $v(r + (s-r)/n) = w_n$ if n is even and $v(r + (s-r)/n) = p$ if n is odd. By Lemma 2.2, there is a function f, from

$[a;b]$ into $\boldsymbol{R}$, continuous on $[a;b]$, such that if n is a positive integer, then $f(v(r+(s-r)/n)) = (sgn(g(w_n)-g(p))/n$ if n is even and $f(v(r+(s-r)/n)) = 0$ of n is odd. Now suppose that D is a subdivision of $[r;s]$ and $0<Q$. There is an even positive integer N such that $r+(s-r)/N<h$, where $[r;h]$ is D. There is an even positive integer $N'>N$ such that

$$c(\sum_{\substack{k \text{ even} \\ N\leq k\leq N'}} 1/k) > Q + [|\sum_{D-\{[r;h]\}} f(v(q))\Delta_{[l;q]}g(v)|]_1 + 2HM$$

where $H = \sup\{|f(x)| : a\leq x\leq b\}$. Let E denote $\{r;r+(s-r)/(N'+1)]$, $[r+(s-r)/(N'+1);\ r+(s-r)/N']$, $\ldots$, $[r+(s-r)/(N+1);\ r+(s-r)/N]$, $[r+(s-r)/N;h]\}$. Clearly, $E\cup(D-\{[r;h]\})\ll D$. Now

$$|\sum_{E\cup(D-\{[r;h]\})} f(v(q))\Delta_{[k;q]}g(v)| \geq |\sum_E f(v(q))\Delta_{[k;q]}g(v)| - [\ ]_1$$

$$= |f(v(r+(s-r)/(N'+1)))(g(v(r+(s-r)/(N'+1))) - g(v(r)))$$

$$+ [f(v(r+(s-r)/N'))(g(v(r+(s-r)/N')) - g(v(r+(s-r)/(N'+1)))) + \ldots$$

$$+ f(v(r+(s-r)/N))(g(v(r+(s-r)/N)) - g(v(r+(s-r)/(N+1))))]_2$$

$$+ f(v(h))(g(v(h)) - g(v(r+(s-r)/N)))| - [\ ]_1 \geq -HM + |[\ ]_2| - HM - |[\ ]_1$$

$$= -2HM - [\ ]_1 + |\sum_{\substack{k \text{ even} \\ N\leq k\leq N'}} f(v(r+(s-r)/k))(g(v(r+(s-r)/k))$$

$$-g(v(r+(s-r)/(k+1))))| = -2HM - [\ ]_1 + \sum_{\substack{k \text{ even} \\ N\leq k\leq N'}} (1/k)|g(w_k) - g(p)|$$

$$\geq -2HM - [\ ]_1 + c(\sum_{\substack{k \text{ even} \\ N\leq k\leq N'}} (1/k)) > Q\,.$$

Thus $\int_r^s f(v(t))\,dg(v(t))$ does not exist, a contradiction. Therefore g is continuous on $[a;b]$.

We now show the second part of the conclusion of (3). We shall carry out a constructive proceedure. So suppose, on the contrary, that there is a function v satisfying the stated conditions such that $g(v)$ does not have bounded variation on

$[r\,;s]$. By routine methods it follows that there is an element k of $[r\,;s]$ such that either $k<s$ and for each y in $(k\,;s]$, $g(v)$ does not have bounded variation on $[k\,;y]$, or $r<k$ and for each y in $[r\,;k)$, $g(v)$ does not have bounded variation on $[y\,;k]$. Without loss of generality we shall assume that former condition. Suppose that $0<K$ and $k<y\leq s$. There is a subdivision D of $[k\,;y]$ such that $\sum_D |\Delta g(v)|>K+M$. We see that D contains at least 2 elements. There is w such that $[k\,;w]$ is in D. $M+\sum_{D-\{[k;w]\}}|\Delta g(v)|\geq\sum_D|\Delta g(v)|>K+M$, so that $\sum_{D-\{[k;w]\}}|\Delta g(v)|>K$. It therefore follows that there is a decreasing sequence $\{l_i\}_{i=1}^{\infty}$ of elements of $(k\,;s]$ such that $l_n\to k$ as $n\to\infty$ and $\sum_{i=1}^{\infty}|g(v(l_{i+1})-g(v(l_i))|\to\infty$ as$n\to\infty$. So far our argument has been almost identical to the corresponding portion of a standard argument for a well-known Stieltjes integrability vs. bounded variation theorem. However, the desire to find, for this situation, a certain element of $C[a\,;b]\boldsymbol{R}$ suggests a modification of v. We proceed accordingly. There are positive number sequences $\{c_i\}_{i=1}^{\infty}$ and $\{d_i\}_{i=0}^{\infty}$ such that $\sum_{i=l}^{\infty}(c_i+d_i)<\infty$ and if $a\leq x\leq y\leq b$, n is a positive integer and $|x-y|<d_n$, then $|g(x)-g(y)|<c_n$. For each positive integer n, let v_n denote the contraction of v to $[l_{n+1}\,;l_n]$. Note, trivially, that if n is a positive integer, then $v_{n+1}(l_{n+1})=v_n(l_{n+1})$. Thus, by induction, Lemma 2.3, and one small routine consideration, there is a sequence $\{v_n^*\}_{n=1}^{\infty}$ such that if n is a positive integer, then v_n^* is an element of $AC[l_{n+1};\ l_n][a\,;b]$, v_n^* and v_n have an element in common, $v_{n+1}^*(l_{n+1})=v_n^*(l_{n+1})$, $v_n^*(l_{n+1})$ is not in $\{v(k),\ v_1^*(l_1),\ \ldots,\ v_n^*(l_n)\}$, $|v_n^*(l_{n+1})-v_n(l_{n+1})|<d_n$, $\int_{[l_{n+1};l_n]}|\mathrm{d}v_n-\mathrm{d}v_n^*|<d_{n-1}$ and $v_1^*(l_1)\neq v(k)$. Now, if m is a positive integer, then

$$\sum_{n=1}^{m}\int_{[l_{n+1};l_n]}|\mathrm{d}v_n^*|\leq\sum_{n=1}^{m}[d_{n-1}+\int_{[l_{n+1};l_n]}|\mathrm{d}v_n|]\leq[\sum_{n=1}^{m}d_{n-1}]+\int_{[l_{m+1};l_1]}|\mathrm{d}v|\leq[\sum_{n=1}^{m}d_{n-1}]+\int_{[r;s]}|\mathrm{d}v|\,.$$

Therefore, by Lemma 2.4, there is an element u of $AC[r\,;s][a\,;b]$ such that $\bigcup_{n=1}^{\infty}v_n^*\subseteq u$.

Now, since for each n, $u(l_n)=v_n^*(l_n)$, it follows that $\{u(l_n)\}_{n=1}^{\infty}$ is a sequence of distinct numbers of $[a\,;b]$, all distinct from $v(k)$ such that $u(l_n)\to u(k)$ as $n\to\infty$. The inequality $|u(l_{p+1})-v(l_{p+1})|=|v_p^*(l_{p+1})-v_p(l_{p+1})|<d_p$ clearly implies that $u(k)=v(k)$, so that each $u(l_n)$ is distinct from $u(k)$; further, this inequality implies

that $|g(u(l_{p+1}))-g(v(l_{p+1}))|<c_p$. Therefore, if m is a positive integer ≥ 2, then

$$\sum_{n=2}^{m}|g(u(1_{n+1}))-g(u(l_n))|$$
$$\geq\sum_{n=2}^{m}[-|g(u(l_{n+1}))-g(v(l_{n+1}))|+|g(v(l_{n+1}))-g(v(l_n))|-|g(v(l_n))-g(u(l_n))|]$$
$$>\sum_{n=2}^{m}-c_n+\sum_{n=2}^{m}|g(v(l_{n+1}))-g(v(l_n))|+\sum_{n=2}^{m}-c_{n-1}\to\infty$$

as $m\to\infty$. Let us note that if $\{a\}_{k=1}^{\infty}$ is a sequence of nonnegative numbers such that $\sum_{k=1}^{n}a_k\to\infty$ as $n\to\infty$, then $\sum_{k=1}^{n}[a_k(1+\sum_{j=1}^{k}a_j)^{-1/2}]\to\infty$ as $n\to\infty$. Now, since $\sum_{i=1}^{n}|g(u(l_{i+1}))-g(u(l_i))|\to\infty$ as $n\to\infty$, it follows that $(1+\sum_{i=1}^{n}|g(u(l_i))-g(u(l_{i+1}))|)^{-1/2}\to 0$ as $n\to\infty$, so that, by Lemma 2.2 there is an element f of $C[a;b]\boldsymbol{R}$ such that if n is a positive integer, then

$$f(u(l_{n+1}))=(1+\sum_{i=1}^{n}|g(u(l_i))-g(u(l_{i+1}))|)^{-1/2}\operatorname{sgn}(g(u(l_n))-g(u(l_{n+1})))\ ,$$

so that if n is a positive integer, then

$$\sum_{j=1}^{n}f(u(l_{j+1}))(g(u(l_j))-g(u(l_{j+1})))$$
$$=\sum_{j=1}^{n}[(1+\sum_{i=1}^{j}|g(u(l_i))-g(u(l_{i+1}))|)^{-1/2}|g(u(l_j))-g(u(l_{j+1}))|]\to\infty$$

as $n\to\infty$. Clearly, to reach a contradiction, it will be enough to show that $\int_k^{l_1}f(u(t))\,\mathrm{d}g(u(t))$ does not exist. Let $W=\sup\{|f(x)|:a\leq x\leq b\}$. Suppose that $0<K$ and D is a subdivision of $[k;l_1]$. There is a refinement D' of D such that for some positive integer N, $[k;l_N]$ is in D'. There is a positive integer $N'>N$ such that

$$\sum_{i=N}^{N'}f(u(l_{i+1}))(g(u(l_i))-g(u(l_{i+1})))>WM+K+\sum_{D'-\{k;l_N]\}}|f(u(q))|\,|g(u(q))-g(u(p))|\ .$$

Let $E=[D'-\{[k;l_n]\}]\cup\{[k;l_{N'+1}],\ [l_{N'+1};l_{N'}],\ \ldots,\ [l_{N+1};l_N]\}$. Clearly E is a refinement of D. Now,

$$\sum_E f(u(p))\,(g(u(q)-g(u(p)))$$

$$=f(u(k))\,(g(u(l_{N'+1}))-g(u(k)))+\sum_{i=N}^{N'} f(u(l_{i+1}))\,(g(u(l_i))-g(u(l_{i+1})))$$

$$+\sum_{D'-\{[k;l_n]\}} f(u(q))\,(g(u(q))-g(u(p)))=f(u(k))\,(g(u(l_{N'+1}))-g(u(k)))$$

$$+\sum_{i=N}^{N'} |f(u(l_{i+1}))|\,|g(u(l_i))-g(u(l_{i+1}))|+\sum_{D'-\{[k;l_N]\}} f(u(q))\,(g(u(q))-g(u(p)))$$

$$\geq -WM+WM+K$$

$$+\sum_{D'-\{[k;l_N]\}} |f(u(q))|\,|g(u(q)-g(u(p))|+\sum_{D'-\{[k;l_N]\}} f(u(q))\,(g(u(q))-g(u(p)))\geq K\,.$$

Therefore $\int_{[k;l_1]} f(u(t))\,\mathrm{d}g(u(t))$ does not exist, and we have a contradiction. Therefore the second part of the conclusion of (3) holds. Therefore (2) implies (3).

We now show that (3) implies (4). Suppose, on the contrary, that (3) is true, but that if $K>0$ and $d>0$, then there is a finite sequence $\{I_k\}_{k=1}^{m}$ of subintervals of $[a;b]$ such that $\sum_{k=1}^{m} \Delta_{I_k} x\leq d$, but $\sum_{k=1}^{m} |\Delta_{I_k} g|>K$.

It follows that for each positive integer n there is a finite sequence $\{I_k^{(n)}\}_{k=1}^{m(n)}$ of subintervals of $[a;b]$ such that $\sum_{k=1}^{m(n)} \Delta_{I_k^{(n)}} x\leq 2^{-(n+2)}$, but $\sum_{k=1}^{m(n)} |\Delta_{I_k^{(n)}} g|>n$; we shall, when appropriate, express $I_k^{(n)}$ explicitly as $[x_k^{(n)};\ y_k^{(n)}]$.

This is to expedite matters: There is a function, δ, from the positive numbers into the positive numbers, such that of $0<c$, each of w and z is in $[a;b]$ and $|w-z|<\delta(c)$, then $|g(w)-g(z)|<c$.

There is a function u_1 in $AC[r;s][a;b]$ such that $u_1(r)=a$ and $u_1(s)=b$.

By Theorem 2.1. and induction there are sequences $\{D_n\}_{n=1}^{\infty}$, $\{D'_n\}_{n=1}^{\infty}$, $\{X_n\}_{n=1}^{\infty}$, $\{u_n\}_{n=1}^{\infty}$, and $\{\{t_k^{(n)}\}_{k=1}^{m(n)}\}_{n=1}^{\infty}$, such that if n is a positive integer, then:

(i)′ $D_n\ll\{[r;s]\}$, $D'_n\subseteq D_n$, X_n is a reversible function from D'_n onto $\{1,\ldots,m(n)\}$, u_n is in $AC[r;s][a;b]$, $u_n(r)=a$ and $u_n(s)=b$.

(ii)′ For all $[p;q]$ in D'_n, $p<t^{(n)}_{X_n[p;q]}<q$ and $D_{n+1}\ll[U_{D'_n}\{[p;t^{(n)}_{X_n[p;q]}],[t^{(n)}_{X_n[p;q]};\ q]\}]\cup[D_n-D'_n]$.

(iii)′ If $[p;q]$ is in D'_n, then $|x^{(n)}_{X_n[p;q]}-u_n(p)|+|u_n(q)-x^{(n)}_{X_n[p;q]}|+\int_{[p;q]}|\mathrm{d}u_n|<\min\{(m(n)\,2^{n+2})^{-1},\ \delta(m(n)^{-1})\}$.

(iv)′ If $[p;q]$ is in D'_n, then $u_{n+1}(p)=u_n(p)$, $u_{n+1}(q)=u_n(q)$, $u_{n+1}(t^{(n)}_{X_n[p;q]}) = y^{(n)}_{X_n[p;q]}$, $\int_{[p;q]} |\mathrm{d}u_{n+1}| = |y^{(n)}_{X_n[p;q]} - u_n(p)| + |y^{(n)}_{X_n[p;q]} - u_n(q)|$, if x is in $[p;q]$ in $D_n - D'_n$, then $u_{n+1}(x)=u_n(x)$ and $\int_{[r;s]} |\mathrm{d}u_n - \mathrm{d}u_{n+1}| \leq \sum_{D'_n} [2|y^{(n)}_{X_n[p;q]} - x^{(n)}_{X_n[p;q]}| + (m(n)\,2^{n+2})^{-1}] < 2(2^{-(n+2)}) + 2^{-(n+2)} < 2^{-n}$.

We now have the following three consequences:

(a) From routine considerations, it follows that there is a function u in $AC[r;s][a;b]$ such that $\int_{[r;s]} |\mathrm{d}u - \mathrm{d}u_n| \to 0$ as $n \to \infty$.

(b) If each of n and n' is a positive integer, $n' > n$, and x is in $U_{D_n}\{p,\ q\}$, then $u_n(x) = u_{n'}(x)$, which implies that $u(x) = u_n(x)$.

(c) If n is a positive integer and $[p;q]$ is in D'_n, then

$$|g(u_{n+1}(t^{(n)}_{X_n[p;q]})) - g(u_{n+1}(p))| + |g(u_{n+1}(t^{(n)}_{X_n[p;q]})) - g(u_{n+1}(q))| \geq |g(y^{(n)}_{X_n[p;q]}) - g(x^{(n)}_{X_n[p;q]})| - |g(x^{(n)}_{X_n[p;q]}) - g(u_{n+1}(p))| + |g(y^{(n)}_{X_n[p;q]}) - g(x^{(n)}_{X_n[p;q]})| - |g(x^{(n)}_{X_n[p;q]}) - g(u_{n+1}(q))| > 2\,|g(y^{(n)}_{X_n[p;q]}) - g(x^{(n)}_{X_n[p;q]})| - 2/m(n)\ .$$

It follows from (b) and (c) above that if n is a positive integer, then

$$\sum_{D'_n} [|g(u(t^{(n)}_{X_n[p;q]})) - g(u(p))| + |g(u(q)) - g(u(t^{(n)}_{X_n[p;q]}))|]$$

$$> \sum_{D'_n} [2\,|g(y^{(n)}_{X_n[p;q]}) - g(x^{(n)}_{X_n[p;q]})| - 2/m(n)] = [2\sum_{D'_n} |g(y^{(n)}_{X_n[p;q]}) - g(x^{(n)}_{X_n[p;q]})|] - 2m(n)/m(n) > 2n - 2\ .$$

This clearly implies that $g(u)$ does not have bounded variation on $[r;s]$, a contradiction. Therefore (3) implies (4).

We now show that (4) implies (5). Suppose that (4) is true. We first show that if each of m and n is a positive integer, I is a subinterval of $[a;b]$ and $\Delta_I x \leq md/n$, then $|\Delta_I g| \leq mK/n$. So suppose that I is a subinterval of $[a;b]$ such that $\Delta_I x \leq md/n$. Then $n\Delta_I x \leq md$. For $I=[w;z]$, let $E = \{[w; w+(z-w)/m], [w+(z-w)/m; w+2(z-w)/m], \ldots, [w+(m-1)(z-w)/m; z]\}$. If $[p;q]$ is in E, then $n\Delta_{[p;q]} x = n\Delta_I x/m \leq md/m = d$, so that $n\,|\Delta_{[p;q]} g| \leq K$. Therefore $n|\Delta_I g| \leq n\sum_E |\Delta_{[p;q]} g| = \sum_E n\,|\Delta_{[p;q]} g| \leq mK$, so that $|\Delta_I g| \leq mK/n$.

Now, again, suppose that I is a subinterval of $[a;b]$. Clearly, $\Delta_I x \leq (\Delta_I x/d)\,d$. There is a decreasing sequence $\{k_n\}_{n=1}^{\infty}$ of rational numbers such that $k_n \to \Delta_I x/d$

as $n \to \infty$. Now, since $\Delta_I x \le k_n d$ for all n, it follows from the preceding paragraph that $|\Delta_I g| \le k_n K$ for all n, so that $|\Delta_I g| \le (\Delta_I x/d) K = (K/d) \Delta_I x$.

Therefore g is Lipschitz on $[a\,;b]$. Therefore (4) implies (5).

Finally we show, quite routinely, that (5) implies (1). Suppose that (5) is true and v in $CBV[r\,;s][a\,;b]$. Clearly, $f(v)$ is continuous on $[r\,;s]$. Let K' denote a Lipschitz constant for g. If $D \ll \{[r\,;s]\}$, then $\sum_D |\Delta g(v)| \le \sum_D K' |\Delta v| \le K' \int_{[r;s]} |dv|$. Therefore $g(v)$ has bounded variation on $[r\,;s]$, so that, as is well known, $\int_{[r;s]} f(v(t))\, dg(v(t))$ exists. Therefore (5) implies (1). Therefore (1), (2), (3), (4) and (5) are equivalent.

We now prove the final statement of the theorem. Suppose that g is Lipschitz on $[a\,;b]$, K' is a Lipschitz constant for g, and f and v are as given in (1). $g(v)$ is in $BV[r\,;s]\boldsymbol{R}$.

Suppose that $0 < c$. There is $d > 0$ such that if each of x and y is in $[a\,;b]$ and $|x-y| < d$, then $|f(x) - f(y)| < c$. There is $D \ll \{[r\,;s]\}$ such that if I is in D and each of w and z is in I, then $|v(w) - v(z)| < d$.

Let u denote the function from $[r\,;s]$ into $\boldsymbol{R}$ such that if t is in $[r\,;s]$, then

$$u(t) = v(r) + \sum_D [(v(q) - v(p))/(q-p)] \max\{\min\{t-p,\ q-p\},\ 0\}\ .$$

Routine considerations tell us that if $[p\,;q]$ is in D, then $u(p) = v(p)$, $u(q) = v(q)$, and that the contraction of u to $[p\,;q]$ is monotonic. It therefore additionally follows that u is a function from $[r\,;s]$ into $[a\,;b]$. We also observe that if $[p\,;q]$ is in D and $p \le x < y \le q$, then $u(y) - u(x) = [(v(q) - v(p))/(q-p)](y-x)$, so that if $p \le h < k \le q$ and $E \ll [h\,;k]$, then $\sum_E |\Delta u| = |(v(q) - v(p))/(q-p)| \sum_E \Delta x$ $= |v(q) - v(p)|\,[(k-h)/(q-p)]$, so that $\int_{[h;k]} |du| = |v(q) - v(p)|\,[(k-h)/(q-p)]$.

Now, suppose that l is u or v. Suppose that $E \ll D$ and for each $[p\,;q]$ in D, $E[p\,;q] = \{J : J \text{ in } E,\ J \subseteq [p\,;q]\}$. If, for each $[h\,;k]$ in E, $z_{[h;k]}$ is in $[h\,;k]$, then, letting $z = z_{[h;k]}$, we have that

$$\Big|\sum_E f(l(z))\,[g(l(k)) - g(l(h))] - \sum_D f(l(p))[g(l(q)) - g(l(p))]\Big|$$

$$= \Big|\sum_D \sum_{E[p;q]} (f(l(z)) - f(l(p)))\,(g(l(k)) - g(l(h)))\Big|$$

$$\le \sum_D \sum_{E[p;q]} |f(l(z)) - f(l(p))|\, |g(l(k)) - g(l(h))| \le \sum_D \sum_{E[p;q]} cK'\, |l(k) - l(h)| \le cK' \sum_D \int_{[p;q]} |dv|\ .$$

It therefore follows that

$$\Big|\int_{[r;s]} f(l(t))\,dg(l(t)) - \sum_D f(l(p))\,(g(l(q)) - g(l(p)))\Big| \le cK' \int_{[r;s]} |dv|\ .$$

Thus

$$\Big|\int_r^s f(v(t))\,dg(v(t)) - \int_{v(r)}^{v(s)} f(x)\,dg(x)\Big|$$

$$= \Big|\int_r^s f(v(t))\,dg(v(t)) - \sum_D f(v(p))\,(g(v(q)) - g(v(p))) + \sum_D f(u(p))\,(g(u(q)) - g(u(p)))$$

$$- \int_r^s f(u(t))\,dg(u(t)) + \int_r^s f(u(t))\,dg(u(t)) - \int_{v(r)}^{v(s)} f(x)\,dg(x)\Big|$$

$$\le cK' \int_{[r;s]} |dv| + cK' \int_{[r;s]} |dv| + 0$$

the value of the last term of this sum a consequence of the definition of u and Corollary 1.1.

Therefore $$\int_r^s f(v(t))\,dg(v(t)) = \int_{v(s)}^{v(r)} f(x)\,dg(x)\ .$$

4 - Transformation of variables and quasi-continuity

In this section we prove Theorem 4.6. as stated in the introduction. We precede the argument, or rather, as we shall see, the indication of the argument, with some preliminary theorems.

Theorem 4.1. *Suppose that* $p<q$ *and* v *is a function whose domain includes* $[p\,;q]$, *whose range* $\subseteq \boldsymbol{R}$, *the contraction of* v *to* $[p\,;q]$ *is in* $C[p\,;q]\boldsymbol{R}$ *and* $|v(x)|>0$ *for all* x *in* $(p\,;q)$. *Then, if* $p<p'<q'<q$, *there is* $d>0$ *such that if* $p\le x\le q$ *and* $|v(x)|<d$, *then* x *is in* $[p\,;p']\cup[q'\,;q]$.

Indication of proof. There is $d>0$ such that if $p'\le x\le q'$, then $d\le|v(x)|$.

Theorem 4.2. *Suppose the hypothesis of Theorem* 4.1. *with the added condition that the contradiction of* v *to* $[p\,;q]$ *is in* $CVB[p\,;q]\boldsymbol{R}$. *Then, if* $0<c$, *then there is* $d>0$ *and* $D\ll\{[p\,;q]\}$ *such that if* $E\ll D$ *and* $E'=\{I:I$ *in* E, $|v(x)|<d$ *for some* x *in* $I\}$ (*if any*), then $\sum_{E'}\int_I |dv|<c$.

Indication of proof. There are p' and q' such that $p<p'<q'<q$ and $\int_{[p;p']}|dv|+\int_{[q';q]}|dv|<c$. By Theorem 4.1. there is $d>0$ such that if $p\le x\le q$ and $|v(x)|<d$, then x is in $[p;p']\cup[q';q]$. Let D denote $\{[p;p'],\ [p';q'],\ [q';q]\}$.

Theorem 4.3. *Suppose that $r<s$, v is in $CVB[r;s]\boldsymbol{R}$, for some x in $[r;s]$, $v(x)\ne 0$, and G is the set to which X belongs iff X a subinterval $[u;w]$ of $[r;s]$ such that if x is in $(u;w)$, then $v(x)\ne 0$ and $[u;w]$ satisfies one of the following consditions*:

(i) *$u=r$ and either $v(w)=0$ or $w=s$*, (ii) *$v(u)=0=v(w)$, or* (iii) *$w=s$ and either $v(u)=0$ or $u=r$. Then* $\int_{[r;s]}|dv|=\sum_G\int_I|dv|$.

Proof. Clearly G is a countable collection of nonoverlapping subintervals of $[r;s]$. We shall take the usual liberties with set intersections and adopt the usual conventions involving variation evaluations. For each subinterval V of $[r;s]$, let $G(V)=\{V\cap I: I \text{ in } G\}$; we see that if $G'(V)$ is a finite subcollection of $G(V)$, then $\sum_{G'(V)}\int_J|dv|\le\int_V|dv|$, so that $\sum_{G(V)}\int_J|dv|\le\int_V|dv|$.

It follows that $\sum_G\int_J|dv|\le\int_{[r;s]}|dv|$. We shall now show that $\int_{[r;s]}|dv|\le\sum_G\int_J|dv|$. For each subinterval V of $[r;s]$, we shall, when feasible, let $\sum_{G(V)}=\sum_{G(V)}\int_J|dv|$ which can also be expressed as $\sum_G\int_{V\cap I}|dv|$.

First suppose that $r\le j<k<l\le s$. We shall show that

$$\sum_{G([j;k])}+\sum_{G([k;l])}\le\sum_{G([j;l])}.$$

There is at most one element W of G overlapping both $[j;k]$ and $[k;l]$; we shall assume the more difficult alternative. Suppose that $0<c$. There are finite subcollections H_1 and H_2 of G, having no element other than W in common, and having W in common iff W overlaps both $[j;k]$ and $[k;l]$, such that no element of H_1 other than W overlaps $[k;l]$, no element of H_2 other than W overlaps $[j;k]$ and $\sum_{G([j;k])}<c/2+\sum_{H_1}\int_{[j;k]\cap I_1}|dv|$ and $\sum_{G([k;l])}<c/2+\sum_{H_2}\int_{[k;l]\cap I_2}|dv|$, so that

$$\sum_{G([j;k])}+\sum_{G([k;l])}<c+\sum_{H_1}\int_{[j;k]\cap I_1}|dv|+\sum_{H_2}\int_{[k;l]\cap I_2}|dv|$$

$$=c+\sum_{H_1\cup H_2}\Big[\int_{[j;k]\cap I}|dv|+\int_{[k;l]\cap I}|dv|\Big]=c+\sum_{H_1\cup H_2}\int_{[j;l]\cap I}|dv|\le c+\sum_{G([j;l])}.$$

Therefore $\sum_{G([j;k])}+\sum_{G([k;l])}\le\sum_{G([j;l])}$.

We now grant that, inductively, it follows that if $D \ll \{[r;s]\}$, then $\sum_D [\sum_{G[p;q]}] \leq \sum_{G[r;s]}$.

Now suppose that $r \leq j < l \leq s$. We shall show that $|v(l)-v(j)| \leq \sum_{G([j;l])}$. If $v(l)=v(j)$, then the inequality follows.

Suppose that $v(l) \neq v(j)$. If $v(j)=0$, then $v(l) \neq 0$ and we leave it to the reader to show that for some x, $j \leq x < l$, $v(x)=0$ and for some I in G, $[x;\ l]=[j;l] \cap I$, so that $|v(l)-v(j)| = |v(l)-v(x)| \leq \int_{[x;l]} |dv| = \int_{[j;l]\cap I} |dv| \leq \sum_{G([j;l])}$. For the case $v(l)=0$, it follows in a similar fashion that $|v(l)-v(j)| \leq \sum_{G([j;l])}$.

Finally, suppose that $0 \neq v(j) \neq v(l) \neq 0$. If, for each x in $(j;l)$, $v(x) \neq 0$, then for some I in G, $[j;l]=[j;l]\cap I$, so that $|v(l)-v(j)| \leq \int_{[j;l]} |dv| = \int_{[j;l]\cap I} |dv| = \sum_{G([j;l])}$. On the other hand, suppose that for some w in $(j;l)$, $v(w)=0$. Again, in a fashion similar to that of the immediately preceding paragraph, with respect to $[j;w]$ and $[w;l]$, it follows that

$$|v(l)-v(j)| \leq |v(l)-v(w)| + |v(w)-v(j)| \leq \sum_{G([w;l])} + \sum_{G([j;w])} \leq \sum_{G([j;l])} .$$

We now see that if $D \ll \{[r;\ s]\}$, then $\sum_D |v(q)-v(p)| \leq \sum_D [\sum_{G([p;q])}] \leq \sum_{G([r;s])} = \sum_G \int_I |dv|$. This implies that $\int_{[r;s]} |dv| \leq \sum_G \int_I |dv|$. Therefore $\int_{[r;s]} |dv| = \sum_G \int_I |dv|$.

Theorem 4.4. *If $r<s$, v is in $CBV[r;s]\boldsymbol{R}$, for some x in $[r;s]$, $v(x) \neq 0$ and $0<c$, then there is $d>0$ and $D \ll \{[r;s]\}$ such that if $E \ll D$ and $E'=\{I:I$ in E, $|v(t)|<d$ for some t in $I\}$ (if any), then $\sum_{E'} \int_I |dv| < c$.*

Proof. For G defined for v, as in Theorem 4.3, $\int_{[r;s]} |dv| = \sum_G \int_I |dv|$, so that there is a finite subcollection G' of G such that $0 \leq \int_{[r;s]} |dv| - \sum_{G'} \int_I |dv| < c/2$. There is a subdivision H of $[r;s]$ including G'. Let N = the number of elements of G'. By Theorem 4.2, for each $[p;q]$ in G', there is $D[p;q] \ll \{[p;q]\}$ and $d_{[p;q]}>0$ such that if $S \ll D[p;q]$ and $S' \subseteq \{I:I$ in S, $|v(x)|<d_{[p;q]}$ for some x in $I\}$ (if any), then $\sum_{S'} \int_I |dv| < c/2N$. Let $D=(H-G') \cup_{G'} D[p;q]$ and $d=\min\{d_{[p;q]}:[p;q]$ in $G'\}$. Suppose that $E \ll D$. For each $[p;q]$ in G' there is $E[p;q] \ll D[p;q]$ and for each $[p;q]$ in $H-G'$ there is $E[p;q] \ll \{[p;q]\}$ such that $E=[\cup_{G'} E[p;q]]$

$\cup [\bigcup_{H-G'} E[p;q]]$. Suppose that $E' = \{I : I \text{ in } E,\ |v(t)| < d \text{ for some } t \text{ in } I\}$ (if any). For each $[p;q]$ in G' there is a subset $E'[p;q]$ of $E[p;q]$ and for each $[p;q]$ in $H-G'$ there is a subset $E'[p;q]$ of $E[p;q]$ such that $E' = [\bigcup_{G'} E'[p;q]] \cup [\bigcup_{H-G'} E'[p;q]]$. It follows that

$$\sum_{E'} \int_J |dv| = \sum_{G'} \sum_{E'[p;q]} \int_J |dv| + \sum_{H-G'} \sum_{E'[p;q]} \int_J |dv| < Nc/2N + \sum_{H-G'} \int_I |dv|$$

$$= c/2 + \int_{[r;s]} |dv| - \sum_{G'} \int_I |dv| < c/2 + c/2 = c\,.$$

Theorem 4.5. *Suppose that $r < s$, $a < b$, v is in $BVC[r;s][a;b]$, each of f and g is a function from $[a;b]$ into $\boldsymbol{R}$ such that g is Lipschitz on $[a;b]$ and there is h in $[a;b]$ such that if $a \le x_1 \le x_2 < h$ or $h < x_1 \le x_2 \le b$, then $f(x_1) = f(x_2)$. Then $\int_r^s f(v(t))\, dg(v(t))$ exists and is $\int_{v(r)}^{v(s)} f(x)\, dg(x)$.*

Proof. There is $K \ge 0$ such that if I is a subinterval of $[a;b]$, then $|\Delta_I g| \le K |\Delta_I x|$. If $v(x) = h$ for all x in $[r;s]$, then the conclusion is immediate. So suppose that for some x in $[r;s]$ $v(x) \ne h$. Suppose that $0 < c$. Letting $v^* = v - h$, we see by Theorem 4.4 that there is $d > 0$ and $D \ll \{[r;s]\}$ such that if $E \ll D$ and $E' = \{I\colon\ I \text{ in } E,\ |v(t) - h| = |v^*(t)| < d \text{ for some } t \text{ in } I\}$ (if any), then $\sum_{E'} \int_I |dv| = \sum_{E'} \int_I |dv^*| < c/16(1+K)(1+M)$, where $M = \max\{|f(x)| : a \le x \le b\}$.

Let $c' = \min\{d,\ c/16(1+K)(1+M)\}$. There is a function, w, in $C[a;b]\boldsymbol{R}$ such that:

(i) If $a < h$, then there is z_1 such that $\max\{a,\ h - c'\} < z_1 < h$ and if $a \le x \le z_1$, then $w(x) = f(x)$.

(ii) If $h < b$, then there is z_2 such that $h < z_2 < \min\{h + c',\ b\}$ and if $z_2 \le x \le b$, then $w(x) = f(x)$.

(iii) If $z_1 \le x \le z_2$, then $|w(x)| \le M$.

Now, suppose that $a \le j < k \le b$, $H^* \ll \{[j;k]\}$ and if I is in H^*, then either $I \subseteq [z_1;\ z_2]$ or I and $[z_1;\ z_2]$ are nonoverlapping. Suppose that $H \ll H^*$. Let $H' = \{I : I \text{ in } H,\ I \subseteq [z_1;\ z_2]\}$ (if any). Then, if l is an interpolating function on H, then

$$|\sum_H f(l_I)\Delta_I g - \sum_H w(l_I)\Delta_I g| \le \sum_{H'} |f(l_I) - w(l_I)|\,|\Delta_I g| + \sum_{H-H'} |f(l_I) - w(l_I)|\,|\Delta_I g|$$

$$\le 2MK2c' + 0 \le 4MKc/16(1+M)(1+K) < c/4.$$

It therefore follows that $|\int_j^k f(x)\,dg(x) - \int_j^k w(x)\,dg(x)| \le c/4$.

By Theorem 3.1, $\int_r^s w(v(t))\,dg(v(t))$ exists and is $\int_{v(r)}^{v(s)} w(x)\,dg(x)$. There is $D' \ll D$ such that if $E \ll D'$ and y is an interpolating function on E, then $|\int_r^s w(v(t))\,dg(v(t)) - \sum_E w(v(y_J))\Delta_J g(v)| < c/4$, so that, letting $E'' = \{J : J \text{ in } E,\ v(y_I)$ in $[z_1; z_2]\}$, we see that $E'' \subseteq E' = \{J : J \text{ in } E,\ |h - v(y_J)| < c' \le d\}$ and consequently

$$|\int_{v(r)}^{v(s)} f(x)\,dg(x) - \sum_E f(v(y_J))\Delta_J g(v)| \le |\int_{v(r)}^{v(s)} f(x)\,dg(x) - \int_{v(r)}^{v(s)} w(x)\,dg(x)| + |\int_{v(r)}^{v(s)} w(x)\,dg(x)$$

$$- \int_r^s w(v(t))\,dg(v(t))| + |\int_r^s w(v(t))\,dg(v(t)) - \sum_E w(v(y_J))\Delta_J g(v)| + |\sum_E [w(v(y_J))$$

$$- f(v(y_J))]\Delta_J g(v)| < c/4 + 0 + c/4 + \sum_{E''} |w(v(y_J)) - f(v(y_J))|\,|\Delta_J g(v)| + \sum_{E-E''} |w(v(y_J))$$

$$- f(v(y_J))|\,|\Delta_J g(v)| \le c/2 + 2MKc/16(1+K)(1+M) + \sum_{E-E''} 0\,|\Delta_J g(v)| < c/2 + c/8 < c\,.$$

Therefore $\int_r^s f(v(t))\,dg(v(t))$ exists and is $\int_{v(r)}^{v(s)} f(x)\,dg(x)$.

Theorem 4.6. *Suppose that $r < s$, $a < b$, v is in $CBV[r;s][a;b]$, each of u and g is a function from $[a;b]$ into $\boldsymbol{R}$, g is Lipschitz on $[a;b]$ and u is quasi-continuous on $[a;b]$. Then $\int_r^s u(v(t))\,dg(v(t))$ exists and is $\int_{v(r)}^{v(s)} u(x)\,dg(x)$.*

Indication of proof. As is well known, u is the uniform limit of a sequence of (finite) linear combinations of functions of the type described in the hypothesis of Theorem 4.5. Further and consequently, $u(v)$ is the uniform limit of a sequence of (finite) linear combinations of functions of the aforementioned type composed with v. It is obvious that $g(v)$ is of bounded variation on $[r;s]$. It is therefore a simple matter to show that Theorem 4.5, together with the above mentioned facts and routine consequences about uniform convergence, integral convergence and integral existence, imply Theorem 4.6; we leave the details to the reader.

References

[1] M. BRUNEAU, *Derivation et integration*, Rend. Circ. Mat. Palermo (2) 28 (1979), 351-376.

[2] C. CARATHEODORY, *Forlesungen über Reele Functionen*, Chelsea Publishing Co., 1948.

[3] N. GUNTHER, *Sur les integrals de Stieltjes et leurs applications aux problemes de la Physique Mathematique*, Chelsea Publishing Co., 1949.

[4] V. A. SKLJARENKO, *Integration by parts in the SCP Burkill integral*, Mat. Sb. (N. S.) 112 (154) (1980), 630-646.

[5] B. S. THOMSON, *On the derived numbers of* VGB_* *functions*, J. London Math. Soc. (2) 22 (1980), 473-485.

Abstract

Suppose that $r<s$, $a<b$ *and* g *is a function from* $[a;\ b]$ *into* $\boldsymbol{R}$. *It is shown, among other things, that the following statements are true.*

(1) *The following two statements are equivalent:*

(i) *If* f *is a function from* $[a;b]$ *into* $\boldsymbol{R}$, *continuous on* $[a;b]$ *and* v *is a function from* $[r;s]$ *into* $[a;b]$, *continuous and of bounded variation on* $[r;s]$, *then* $\int_r^s f(v(t))\mathrm{d}g(v(t))$ *exists;* (ii) g *satisfies a Lipschitz condition on* $[a;b]$.

(2) *If* g *satisfies* (ii) *above,* v *is as given in* (i) *and* u *is a function from* $[a;b]$ *into* $\boldsymbol{R}$, *quasi-continuous on* $[a;b]$, *then* $\int_r^s u(v(t))\,\mathrm{d}g(v(t))$ *exists and is* $\int_{v(r)}^{v(s)} u(x)\,\mathrm{d}g(x)$.

***