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Some remarks on asymptotically almost automorphic functions (**)

1 - Introduction

Let be X a Banach space and f: $\mathbb{R}^+ \to X$ a (strongly) continuous function. Then we say f is *asymptotically almost automorphic* (a.a.a.) if $f(t) = g(t) + h(t)$, $t \in \mathbb{R}^+$ where g: $\mathbb{R} \to X$ is almost automorphic and h: $\mathbb{R}^+ \to X$ is a (strongly) continuous function such that $\lim_{t\to\infty} h(t) = 0$ (null vector). g is called the *principal term* of f and h its corrective term. For more details on a.a.a. functions, see $[2]_{1,2}$.

A function φ: $\mathbb{R} \to X$ is a *complete trajectory* of a C_0-semigroup $T(t)$, $t \in \mathbb{R}^+$, if $\varphi(t)$ verifies the functional equation $\varphi(t) = T(t-a)\, \varphi(a) \quad \forall a \in \mathbb{R} \quad \forall t \geqslant a$.

If for a C_0-semigroup $T(t)$, $t \in \mathbb{R}^+$, there exists $e \in X$ such that $T(t)\,e$: $\mathbb{R}^+ \to X$ is an asymptotically almost periodic function, then the principal term of $T(t)\,e$ is a complete trajectory of $T(t)$. This result has been proved recently by S. Zaidman (see [3] Theorem 2). Here we generalize it to a.a.a. case.

2 – Theorem. *Suppose $T(t)\,e$ is a.a.a. for some $e \in X$; then its principal term is a complete trajectory of $T(t)$.*

Proof. Let be $T(t)\,e = g(t) + h(t)$ where g and h are resptectively the principal term and the corrective term of $T(t)\,e$. Then there exists a subsequence

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$(n_k)_{k=1}^{\infty}$ of $(n)_{n=1}^{\infty}$ such that

$$\lim_{k\to\infty} g(t+n_k) j(t) \text{ for each } t \in \mathrm{R} \qquad \lim_{k\to\infty} j(t-n_k) = g(t) \text{ for each } t \in \mathrm{R}.$$

Put $\varphi(t) = T(t)\, e$. Then $\varphi(0) = e$. Let us fix $a \in \mathrm{R}$ and take k large enough, such that $a + n_k \geqslant 0$. If $s \geqslant 0$, then

$$\varphi(a+s+n_k) = T(a+s+n_k)\,\varphi(0) = T(s)\,T(a+n_k)\,\varphi(0) = T(s)\,\varphi(a+n_k)\ .$$

Therefore we have

$$g(a+s+n_k) + h(a+s+n_k) = T(s)\,\varphi(a+n_k) \quad \text{where } s \geqslant 0 \quad a+n_k \geqslant 0\ .$$

On the other hand we have

$$\lim_{k\to\infty} g(a+s+n_k) = j(a+s) \qquad \lim_{k\to\infty} h(a+s+n_k) = 0$$

and
$$\lim_{k\to\infty} \varphi(a+s+n_k) = \lim_{k\to\infty} T(s)\,\varphi(a+n_k) = j(a+s)\ .$$

We also have
$$\lim_{k\to\infty} \varphi(a+n_k) = j(a)\ .$$

Therefore, using continuity of $T(s)$,

$$\lim_{k\to\infty} T(s)\,\varphi(a+n_k) = T(s)\, j(a)\ .$$

We can establish the following equality

$$T(s)\, j(a) = j(a+s) \qquad a \in \mathrm{R} \qquad s \geqslant 0\ .$$

But

$$\lim_{k\to\infty} j(t-n_k) = g(t) \quad \forall\, t \in \mathrm{R} \qquad j(a-n_k+s) = T(s)\, j(a-n_k) \quad a \in \mathrm{R} \quad s \geqslant 0\,.$$

Therefore

$$\lim_{k\to\infty} j(a-n_k+s) = T(s)\, g(a) \qquad a \in \mathrm{R} \qquad s \geqslant 0,$$

$$g(a+s) = T(s)\, g(a) \qquad a \in \mathrm{R} \qquad s \geqslant 0.$$

Finally, let us put $s = t - a$ with $t \geqslant a$. Then

$$g(t) = T(t-a)\,g(a) \qquad t \geqslant a \qquad a \in \mathbb{R}.$$

3 – Remark. If $f\colon \mathbb{R}^+ \to X$ is a.a.a., then for any sequence $(s'_n)_{n=1}^{\infty}$ such that $s'_n > 0$ $\forall n$ and $\lim\limits_{n\to\infty} s'_n = +\infty$, we can extract a subsequence $(s_n)_{n=1}^{\infty}$ such that the sequence $(f(t+s_n))_{n=1}^{\infty}$ converges for each $t \in \mathbb{R}^+$. This is well known for asymptotically almost periodic functions (see [1] for example for numerical functions). The proof is obvious, using the definition of a.a.a. functions.

References

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