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On semi-invariant submanifolds of Kaehlerian manifolds (**)

Introduction

The purpose of the present paper is to extend some results obtained about invariant submanifolds [3], [8], [12] and about complex hypersurfaces [11] of Kaehlerian manifolds to semi-invariant submanifolds. There are many interesting examples of submanifolds of this kind. For example, every orientable hypersurface of an almost Hermitian manifold (so, the sphere and the tangent sphere bundle are of this kind); the fibers of certain fiber bundles; and some generalized Heisenberg groups.

In 1 we recall the definition of semi-invariant submanifolds of an almost complex manifold giving also some relations that will be used later. In 2 some properties of semi-invariant submanifolds immersed in a Kaehlerian manifold are considered and a condition to be totally geodesic is also given.

In 3 and 4 we study respectively the cases where the normal connection of the submanifold is trivial and where the ambient manifold has constant holomorphic sectional curvature $\tilde{c}$. In this later paragraph we show that the following conditions are equivalent: (i) the submanifold has flat normal connection; (ii) the Ricci tensors of the manifold and of the submanifold coincide; (iii) $\tilde{c} = 0$ and the submanifold is totally geodesic; (iv) the Ricci tensor of the submanifold vanishes.

Some examples are given in each paragraph and we end the paper giving in 5 further examples of semi-invariant submanifolds immersed in almost complex manifolds.

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1 - Semi-invariant submanifolds of almost complex manifolds

Let M be a submanifold immersed in an almost complex manifold $\tilde{M}$ with almost complex structure J. Let U be a non-vanishing vector field on $\tilde{M}$, never tangent to M, such that

$$\text{(i)} \quad JU = \xi \in \mathfrak{X}(M) \qquad \text{(ii)} \quad JX = \varphi X - \eta(X)\, U \quad \forall X \in \mathfrak{X}(M)$$

where φX is the tangential component to M of JX and η a 1-form on M. In these conditions we say that M is a *semi-invariant submanifold* of $\tilde{M}$ with respect to U.

It is well known that any semi-invariant submanifold of an almost complex manifold is an almost contact manifold with the induced structure (φ, ξ, η). Furthermore, if $\tilde{M}$ is an almost Hermitian manifold with structure $(J, \bar{g})$ and U is a unit vector field normal to M then (φ, ξ, η, g) is an almost contact metric structure on M, where g is the induced metric [1]. Hereafter, $\tilde{M}$ denotes an almost Hermitian manifold and M a semi-invariant submanifold of $\tilde{M}$.

The following lemmas are of easy proof.

Lemma 1.1. *Let U a non-vanishing vector field normal to M. Then the following properties hold*: (i) *if U is Killing, then U is autoparallel*; (ii) *if U is Killing and parallel in the normal bundle, then U is parallel on M*, i.e., $\bar{\nabla}_X U = 0 \quad \forall X \in \mathfrak{X}(M)$, *where $\bar{\nabla}$ denotes the Riemannian connection of the metric $\bar{g}$.*

Lemma 1.2. *Let U be a non-vanishing vector field normal to M and parallel on M and let $B(X, Y)$ denote the second fundamental form of the submanifold. Then*: (i) *$B(X, Y)$ has no component in the direction of U*; (ii) *$JB(X, Y)$ has neither tangential component to M nor component in the direction of U.*

Examples. (1) Let M be an orientable hypersurface of an almost Hermitian manifold $\tilde{M}$, and let U be the unit vector field normal to M. Then, $JU = \xi$ is tangent to M and $JX = \varphi X - \eta(X)\, U$ for all $X \in \mathfrak{X}(M)$, and clearly M is a semi-invariant submanifold of $\tilde{M}$ with respect to U. In particular, S^{2n+1} is a semi-invariant submanifold of $\mathbb{C}^{n+1}$ with respect to the unit normal vector field $U_z = (z_1, \ldots, z_{n+1})$ for all $z = (z_1, \ldots, z_{n+1}) \in S^{2n+1}$.

(2) Let $P(\mathbb{C}^n)$ be the complex projective space and π: $S^{2n+1} \to P(\mathbb{C}^n)$ the canonical projection. For $r > 0$ we consider the hypersurface of S^{2n+1} given by

$$M'(2n,\ r) = \{(z_1,\ \dots,\ z_{n+1}) \in S^{2n+1} \,|\, \sum_{i=1}^{n} |z_i|^2 = r|z_{n+1}|^2\}$$

and for $m \in \mathbb{N}$ such that $2 \leq m \leq n-1$ $(n \geq 3)$ and $s > 0$, we consider

$$M'(2n,\ m,\ s) = \{(z_1,\ \dots,\ z_{n+1}) \in S^{2n+1} \,|\, \sum_{i=1}^{m} |z_i|^2 = s \sum_{i=m+1}^{n+1} |z_i|^2\} .$$

Then, $\pi(M'(2n,\ r)) = M(2n-1,\ r)$ and $\pi(M'(2n,\ m,\ s)) = M(2n-1,\ m,\ s)$ are connected compact hypersurfaces of $P(\mathbb{C}^n)$ and, by consequence, semi-invariant submanifolds of it.

(3) Let M be a Riemannian manifold with local coordinates (x^i) $(i = 1,\ \dots,\ n)$ and TM its tangent bundle. It is well known [5] that TM admits an almost Hermitian structure J. Then, the tangent sphere bundle $T_1 M$ is a semi-invariant submanifold of TM with respect to the unit normal vector field U given at a point $Z = y^i\, \partial/\partial x^i \in T_1 M$ by $U_z = y^i(\partial/\partial x^i)^V$.

Note that the examples (1) and (2) are in fact submanifolds immersed in a Kaehlerian manifold. So, we study this situation in the next paragraph.

2 - Semi-invariant submanifolds of Kaehlerian manifolds

Let $\tilde{M}$ denote a Kaehlerian manifold with the almost Hermitian structure $(J,\ \tilde{g})$ and M a semi-invariant submanifold of $\tilde{M}$ with respect to a unit vector field U normal to M. From the definition of Kaehlerian structure it follows that if U is parallel on M then the almost contact metric structure $(\varphi,\ \xi,\ \eta,\ g)$ on M satisfies

$$(2.1) \qquad (\nabla_x \varphi) Y - ((\nabla_x \eta) Y) U + B(X,\ \varphi Y) + JB(X,\ Y) = 0$$

for all $X,\ Y \in \mathcal{X}(M)$.

Proposition 2.1. *If the vector field U is parallel on M then the almost contact metric structure $(\varphi,\ \xi,\ \eta,\ g)$ is a cosymplectic structure on M. Furthermore, M is minimal.*

Proof. The first part is obtained considering in (2.1) the tangential component to M, the normal component to M in the direction of U and the normal component to M not in the direction of U, taking into account Lemma 1.2. From the last component, changing X by φX, we get

$$B(\varphi X, \varphi Y) = -B(X, Y) \quad \forall X, Y \in \mathfrak{X}(M). \tag{2.2}$$

Hence, M is minimal.

Let $\tilde{R}$ (resp. R) denote the curvature tensor of $\tilde{M}$ (resp. M) and consider a plane section on $\tilde{M}$ spanned by an orthonormal basis $\{\tilde{X}, J\tilde{X}\}$. The *holomorphic sectional curvature* is defined by $\tilde{R}(\tilde{X}, J\tilde{X}, \tilde{X}, J\tilde{X})$. Analogously, if we take a φ-section on M spanned by an orthonormal basis $\{X, \varphi X\}$ with X orthogonal to ξ, the *φ-sectional curvature* is defined by $R(X, \varphi X, X, \varphi X)$. Using the Gauss equation and (2.2) we get that the relation between them is given by

$$\tilde{R}(X, JX, X, JX) = R(X, \varphi X, X, \varphi X) + 2\tilde{g}(B(X, X), B(X, X)) \tag{2.3}$$

for all X unit vector field tangent to M and orthogonal to ξ.

The next consequence is immediate.

Proposition 2.2. *M is totally geodesic if and only if*

$$\tilde{R}(X, JX, X, JX) = R(X, \varphi X, X, \varphi X)$$

for all X unit vector field tangent to M and orthogonal to ξ.

Theorem 2.3. *Let $\tilde{M}$ be of constant holomorphic sectional curvature $\tilde{c}$ and dimension $2(n+p+1)$, $p < n(n+1)/2$. Let M be of dimension $2n+1$ and U parallel on M. In these conditions, M is totally geodesic if and only if M has constant φ-sectional curvature c. Each of these conditions yield $c = \tilde{c}$.*

Proof. If M is totally geodesic, by (2.3) it is obvious that M has constant φ-sectional curvature $c = \tilde{c}$.

Suppose M has constant φ-sectional curvature c. If $c = \tilde{c}$ then by (2.3) $B(X, Y) = 0$ because the second fundamental form is a symmetric bilineal form. Let us now assume $c \neq \tilde{c}$. Given an orthonormal basis $\{E_i, \varphi E_i, \xi\}$ $(1 \leq i \leq n)$ of $T_X(M)$ it is well known that the $n(n+1)$ vectors $B(E_i, E_j)$, $JB(E_i, E_j)$

$(1 \leq i \leq j \leq n)$ are linearly independent (see [6], [13]). Moreover, U is orthogonal to all these vectors by Lemma 1.2. Hence there are at least $n(n+1)+1$ linearly independent vectors in the orthogonal complement of $T_X(M)$ which has dimension $2p+1$, contradicting the hypothesis $p < n(n+1)/2$. Then, $c = \tilde{c}$ and M is totally geodesic.

It is well known [5] that the standard almost Hermitian structure on TM is a Kaehlerian structure if and only if the metric g on M is flat. In this case, the tangent sphere bundle T_1M is a semi-invariant submanifold of a Kaehlerian manifold.

Now, we consider a fiber bundle $P(M, G, \pi)$, M being a manifold with an almost contact structure (φ, ξ, η) and G an odd-dimensional connected Lie group with a left invariant almost contact structure $(\hat{\varphi}, \hat{\xi}, \hat{\eta})$. Let ω be a 1-form of connection. Then P admits an almost complex structure J defined by

$$\pi(JX) = \varphi(\pi X) + \hat{\eta}(\omega X)\,\xi \qquad \omega(JX) = \hat{\varphi}(\omega X) - \eta(\pi X)\,\hat{\xi}\,.$$

Then the fibers of $P(M, G, \pi)$ are semi-invariant submanifolds of P with respect to $U = -\xi^H$, and if the curvature of the connection is zero then U is parallel along the fibers. Furthermore, if the curvature is zero and if the almost contact structures on M and G are cosymplectic, then the almost complex structure J is Kaehlerian.

On the other hand, we remark that every hypersurface of an almost complex manifold has flat normal connection and then the previous examples are submanifolds with flat normal connection immersed in a Kaehlerian manifold. We study this situation in the next paragraph.

3 - Semi-invariant submanifolds of Kaehlerian manifolds with trivial normal connection

$\tilde{M}$ denotes, as before, a Kaehlerian manifold with structure tensors $(J, \tilde{g})$ and M a semi-invariant submanifold of $\tilde{M}$ with respect to U, where U is a unit vector field normal to M and parallel on M. Using Gauss-Weingarten equations, Lemma 1.2 and (2.2), we have that for all vector field $\tilde{N}$ normal to M

$$\varphi A_{\tilde{N}} = \varphi A_N = A_{JN} = -A_N\,\varphi = -A_{\tilde{N}}\,\varphi \tag{3.1}$$

where $\tilde{N}=N+\lambda U$, N orthogonal to U, and $A_{\tilde{N}}$ is the second fundamental tensor of M, which is related with $B(X, Y)$ by $g(A_{\tilde{N}}X, Y)=\tilde{g}(B(X, Y), \tilde{N})$.

Now, let $\tilde{N}$ be a parallel normal section in the normal bundle to M. Then $\tilde{g}(R^{\perp}(X, Y)\tilde{N}, \tilde{N}')=0$ for all X, Y vector fields tangent to M and for all $\tilde{N}'$ normal to M.

By using the Ricci equation we get

$$\tilde{g}(\tilde{R}(X, Y)\tilde{N}, \tilde{N}')=g([A_{N'}, A_N]X, Y) \tag{3.2}$$

where $\tilde{N}=N+\lambda U$, $\tilde{N}'=N'+\mu U$.

In particular, taking $\tilde{N}'=JN$ in (3.2), using (3.1) and substituting Y by φX we conclude

$$\tilde{R}(JN, \tilde{N}, X, \varphi X)=2g(A_N\varphi X, A_N\varphi X)\,.$$

As a consequence, the holomorphic bisectional curvature for the plane sections $\{X, JX\}$ (where X is tangent to M and orthogonal to ξ) and $\{\tilde{N}, J\tilde{N}\}$ (where $\tilde{N}=N+\lambda U$ is normal to M and parallel in the normal bundle) is given by

$$\begin{aligned} H(X, \tilde{N}) &= \tilde{R}(X, JX, \tilde{N}, J\tilde{N})=-\tilde{R}(JN, \tilde{N}, X, \varphi X)+\lambda\tilde{R}(\tilde{N}, \xi, X, \varphi X) \\ &= -2g(A_N\varphi X, A_N\varphi X)\leq 0\,. \end{aligned} \tag{3.3}$$

Assuming now that the normal connection of M is trivial (i.e. $R^{\perp}=0$) we can derive new results. The next lemma [2] will be useful

Lemma 3.1. *Let M be an m-dimensional submanifold of an n-dimensional Riemannian manifold N. Then, the normal connection of M is trivial if and only if there exist $n-m$ orthogonal normal unit vector fields $\{F_1, \ldots, F_{n-m}\}$ such that F_i $(1\leq i\leq n-m)$ is parallel in the normal bundle.*

Therefore, if $\{E_i, \varphi E_i, \xi\}$ $(i=1, \ldots, n)$ is a φ-basis defined on an open subset of M and M has trivial normal connection we can complete this basis with parallel normal vector fields in the normal bundle and take a J-basis of the form $\{E_i, F_j, U, JE_i, JF_j, \xi\}$ $(i=1, \ldots, n,\ j=1, \ldots, p)$. This is possible because, under the hypothesis that U is parallel on M, if F_j is parallel in the normal bundle then JF_j is also parallel.

Proposition 3.2. *If the holomorphic bisectional curvature is non-negative and M has trivial normal connection, then M is a totally geodesic submanifold of $\tilde{M}$.*

Proof. Let X be a unit vector field tangent to M and orthogonal to ξ. Taking $\tilde{N} = F_j$ in (3.3) and from the hypothesis we get that $A_{F_j}\varphi X = 0$ $(j = 1, \ldots, p)$. Thus

$$\tilde{g}(B(\varphi X, \varphi Y), F_j) = g(A_{F_j}\varphi X, \varphi Y) = 0 \quad (j = 1, \ldots, p) \quad \forall X, Y \in \mathfrak{X}(M) .$$

Analogously, $\tilde{g}(B(\varphi X, \varphi Y), JF_j) = 0$ $(j = 1, \ldots, p)$. And then, by (2.2) and Lemma 1.2,

$$B(X, Y) = -B(\varphi X, \varphi Y) = 0 \quad \forall X, Y \in \mathfrak{X}(M) .$$

Let $\tilde{S}$ (resp. S) denote the Ricci tensor of $\tilde{M}$ (resp. M). Using (2.2), (3.1), Lemma 1.2, the Gauss equation and the first Bianchi identity we have that the relation between the Ricci tensors of $\tilde{M}$ and M is given by

$$\tilde{S}(X, Y) \tag{3.4}$$
$$= S(X, Y) - \eta(X) \sum \tilde{R}(U, Y, JF_j, F_j) + \tilde{R}(U, Y, U, X) \quad \forall X, Y \in \mathfrak{X}(M) .$$

And also

Theorem 3.3. *Let $\tilde{M}$ a Kaehlerian manifold, M a semi-invariant submanifold of $\tilde{M}$ with respect to U, where U is a Killing unit normal vector field to M and parallel in the normal bundle. Then, if M has trivial normal connection, the Ricci tensors of $\tilde{M}$ and M coincide.*

Proof. Using Lemma 1.1 we can write $\tilde{R}(U, Y, U, X) = 0$. In a similar way, by using (3.1), the Ricci equation and the triviality of the normal connection we have $\tilde{R}(U, Y, JF_j, F_j) = 0$.

The theorem follows by substitution of this into (3.4).

4 - Semi-invariant submanifolds of complex space forms

Hereafter, $\tilde{M}(\tilde{c})$ denotes a complex space form, i.e., a Kaehlerian manifold with constant holomorphic sectional curvature $\tilde{c}$ and M a semi-invariant submanifold of $\tilde{M}$ with respect to U, where U is a unit vector field normal to M and parallel on M.

It is well known [10] that in this case the curvature tensor of $\tilde{M}$ satisfies

$$\tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z} = \frac{\tilde{c}}{4}[\tilde{g}(\tilde{Y}, \tilde{Z})\tilde{X} - \tilde{g}(\tilde{X}, \tilde{Z})\tilde{Y} + \tilde{g}(\tilde{Z}, J\tilde{Y})J\tilde{X} - \tilde{g}(\tilde{Z}, J\tilde{X})J\tilde{Y} + 2\tilde{g}(\tilde{X}, J\tilde{Y})J\tilde{Z}]\,. \tag{4.1}$$

From the Gauss-Weingarten formulas, the normal component of the curvature tensor is given by

$$(\tilde{R}(X, Y)\tilde{N})^{\perp} = R^{\perp}(X, Y)\tilde{N} - B(A_{\tilde{N}}Y, X) + B(A_{\tilde{N}}X, Y) \tag{4.2}$$

for $X, Y \in \mathcal{X}(M)$ and $\tilde{N} \in \mathcal{X}(M)^{\perp}$.

In addition, if the normal connection of M is trivial we have

$$(\tilde{R}(X, Y)\tilde{N})^{\perp} = B(A_{\tilde{N}}X, Y) - B(A_{\tilde{N}}Y, X)\,. \tag{4.3}$$

Let $\tilde{N} = N + \lambda U$ be a normal section to M with N orthogonal to U. By (4.1) we obtain

$$\tilde{R}(X, Y)\tilde{N} = \frac{\tilde{c}}{4}[\lambda(\eta(X)\varphi Y - \eta(Y)\varphi X + 2g(X, \varphi Y)\xi) + 2g(X, \varphi Y)JN]\,. \tag{4.4}$$

Using (3.1), (4.3) and (4.4) we have the following relation holds

$$\frac{\tilde{c}}{4}[\eta(X)\eta(Y) - g(X, Y)]\tilde{g}(N, N) = g(A_{\tilde{N}}X, A_{\tilde{N}}Y)\,. \tag{4.5}$$

Proposition 4.1. *If M has trivial normal connection then $\tilde{c} \leq 0$. Furthermore, $\tilde{c} = 0$ if and only if M is totally geodesic.*

Proof. It follows from (4.5).

Now, let $\tilde{N}$ be a unit vector in $T_m(M)^{\perp}$. $A_{\tilde{N}}$ is an endomorphism of $T_m(M)$. We denote by $E(\tilde{N})$ the subspace of $T_m(M)$ spanned by all vectors $A_{\tilde{N}}X$, $X \in T_m(M)$. $T_m(M)$ has dimension $2n+1$, therefore $\dim E(\tilde{N}) \leq 2n+1$. But if ξ is a vector of an orthogonal basis $\{E_1, \ldots, E_{2n}, \xi\}$ of $T(M)$, then $A_{\tilde{N}}\xi = 0$ and hence $\dim E(\tilde{N}) \leq 2n$.

Proposition 4.2. *Let $\tilde{M}(\tilde{c})$ be a complex space form, M a semi-invariant submanifold of $\tilde{M}$ with respect to U and with trivial normal connection, U being parallel on M. If there exists a vector $N \in T_m(M)^{\perp}$, $N \neq U$, such that $\dim E(N) \neq 2n$, then M is totally geodesic.*

Proof. Using (4.5) replacing X and Y by the vectors E_i, E_j $(i \neq j)$ of the precedent orthogonal basis we get $g(A_N E_i, A_N E_j) = 0$ $(i, j = 1, \ldots, 2n, i \neq j)$. If $A_N E_i \neq 0$ $(i = 1, \ldots, 2n)$ there are $2n$ non-vanishing orthogonal vectors in $E(N)$. Hence $\dim E(N) = 2n$, contradicting the hypothesis. Thus, $A_N E_j = 0$ for some j, and, by using (4.5), we get

$$0 = g(A_N E_j, A_N E_j) = -\frac{\tilde{c}}{4}\,.$$

We conclude that $\tilde{c} = 0$ and, according to Proposition 4.1, M is totally geodesic.

On the other hand, if we take in (4.5) $\tilde{N} = N + N'$ with N and N' orthogonal normal vector field on M and orthogonal to U, we get

$$A_N A_{N'} + A_{N'} A_N = 0 \tag{4.6}$$

and we have

Theorem 4.3. *Let $\tilde{M}(\tilde{c})$ be a complex space form of dimension $2(n+p+1)$, $p \geq 2$. Let M be a $(2n+1)$-dimensional semi-invariant submanifold with respect to U, U being parallel on M. Then, the following conditions are equivalent*:

(i) *M has trivial normal connection.* (ii) *$\tilde{c} = 0$ and M is totally geodesic.*

Proof. (i $\Rightarrow$ ii) Assume that M is not totally geodesic. From Proposition 4.2 the dimension of $E(N)$ is $2n$ for every $N \in T_m(M)^{\perp}$, $N \neq U$. Since $p \geq 2$, $\dim T_m(M) \geq 5$ and there is at least another vector N' orthogonal to N such that the vectors N, N', JN, JN', U belong to an orthogonal basis of $T_m(M)^{\perp}$. From

(4.3) and (4.6) we get

$$\tilde{g}((\tilde{R}(X, Y)N)^{\perp}, N') = 2g(A_N X, A_{N'} Y) . \tag{4.7}$$

On the other hand, by using (4.1) we have

$$\tilde{g}((\tilde{R}(X, Y)N)^{\perp}, N') = \frac{\tilde{c}}{2} g(X, \varphi Y)\tilde{g}(JN, N') = 0 . \tag{4.8}$$

Then, from (4.7) and (4.8), $g(A_N X, A_{N'} Y) = 0$ for $X, Y \in T_m(M)$. Hence, the subspace $E(N)$ is orthogonal to $E(N')$. Therefore, $\dim(E(N) \cup E(N')) = 4n$, in contradiction with $E(N) \cup E(N') \subseteq T_m(M)$ and $\dim T_m(M) = 2n+1$. Consequently, M is totally geodesic and, from Proposition 4.1, $\tilde{c} = 0$.

(ii $\Rightarrow$ i) follows directly from (4.1) and (4.2).

If $p = 1$ we need to impose a stronger condition on U:

Theorem 4.4. *Under the hypothesis of the precedent theorem, if $p = 1$ and U is Killing and parallel in the normal bundle, then M has trivial normal connection if and only if M is totally geodesic.*

Now, suppose $\tilde{M}(\tilde{c})$ is a $2(n+p+1)$-dimensional manifold and M is a $(2n+1)$-dimensional submanifold. It is well known [7] that the Ricci tensor $\tilde{S}$ of $\tilde{M}$ is given by

$$\tilde{S}(\tilde{X}, \tilde{Y}) = \frac{\tilde{c}}{2}(n+p+2)\tilde{g}(\tilde{X}, \tilde{Y}) \quad \text{for } \tilde{X}, \tilde{Y} \in \mathfrak{X}(\tilde{M}) . \tag{4.9}$$

Using (2.3), (4.1) and the Gauss equation we deduce that the Ricci tensor S of M is given by

$$S(X, Y) = \frac{\tilde{c}}{4}(2n+3)g(X, Y) - \frac{3\tilde{c}}{4}\eta(X)\eta(Y) - 2\sum \tilde{g}(B(X, E_i), B(Y, E_i)) \tag{4.10}$$

for $X, Y \in \mathfrak{X}(M)$. Then

Theorem 4.5. *Let $\tilde{M}(\tilde{c})$ be a complex space form of dimension $2(n+p+1)$, $p \geq 2$. Let M be a semi-invariant submanifold of dimension $2n+1$*

with respect to U, U being a unit vector field normal to M and parallel on M. Then the following conditions are equivalent:

(i) *M has trivial normal connection.*

(ii) *$\tilde{S}(X, Y) = S(X, Y)$ for $X, Y \in \mathfrak{X}(M)$.*

(iii) *$\tilde{c} = 0$ and M is totally geodesic.*

(iv) *$S(X, Y) = 0$ for $X, Y \in \mathfrak{X}(M)$.*

Proof. (i$\Leftrightarrow$iii) was already proved in Theorem 4.3. (ii$\Rightarrow$iii) Equalling the right sides of both (4.9) and (4.10) and taking $X = Y = \xi$, we obtain $\tilde{c} = 0$. Putting this value in (4.9) and (4.10) with $Y = X$ we have $\sum \tilde{g}(B(X, E_i), B(X, E_i)) = 0$. Hence, $B(X, E_i) = 0$ ($i = 1, \ldots, n$) and M is totally geodesic. (iii$\Rightarrow$iv) follows directly from (4.10). (iv$\Rightarrow$ii) Taking $X = Y = \xi$ in (4.10) we get $\tilde{c} = 0$, and then $\tilde{S}(X, Y) = S(X, Y)$.

Corollary 4.6. *Under the hypothesis of the precedent theorem, if M has trivial normal connection then M is flat.*

On the other hand, if M has constant φ-sectional curvature, the following equalities hold [9] for all $X, Y, Z, V \in \mathfrak{X}(M)$

$$(4.11) \qquad R(V, Z, X, Y)$$

$$= \frac{c}{4}[g(X, V)g(Y, Z) - g(X, Z)g(Y, V)] + \frac{c}{4}[\eta(X)\eta(Z)g(V, Y) + \eta(V)\eta(Y)g(X, Z) - \eta(X)\eta(V)g(Y, Z) - \eta(Y)\eta(Z)g(X, V) + \Phi(X, Z)\Phi(V, Y) - \Phi(X, V)\Phi(Z, Y) + 2\Phi(X, Y)\Phi(V, Z)]$$

where Φ denotes the fundamental 2-form of the cosymplectic structure.

$$(4.12) \qquad S(X, Y) = \frac{(n+1)c}{2} g(X, Y) - \frac{(n+1)c}{2} \eta(X)\eta(Y) .$$

And we can state the following

Theorem 4.7. *Let $\tilde{M}(\tilde{c})$ be a complex space form. Let M be a semi-*

invariant submanifold with respect to U, U being parallel on M, with constant φ-sectional curvature. Then, the following properties hold:

(i) $\tilde{c}=0$ *and* $c\leq 0$.

(ii) *M is locally symmetric* ($\nabla R=0$).

(iii) *The Ricci tensor of M is parallel* ($\nabla S=0$).

Proof. (i) follows equalling the right sides of both (4.10) and (4.12) with $X=Y=\xi$. (ii) and (iii) can be proved after some lengthy calculation taking into account (4.11), (4.12) and Proposition 2.1.

5 - Further examples

Let M and M' be two manifolds with almost contact metric structures given respectively by (φ, ξ, η, g) and $(\varphi', \xi', \eta', g')$. We consider the Riemannian product $\tilde{M}=M\times M'$, with the metric

$$\tilde{g}((X, X'), (Y, Y'))=g(X, Y)+g'(X', Y')$$

and the almost complex structure J defined by

$$J(X, X')=(\varphi X+\eta'(X')\xi, \varphi' X'-\eta(X)\xi') .$$

If the almost contact metric structures (φ, ξ, η, g) and $(\varphi', \xi', \eta', g')$ are cosymplectic, then $\tilde{M}$ is a Kaehlerian manifold and M (naturally immersed in $\tilde{M}$) is a semi-invariant submanifold of $\tilde{M}$ with respect to the vector field $U=(0, \xi')$, which is parallel on M.

Let $H(p, r)$ the generalized Heisenberg group, that is, the Lie group of real matrices of the form

$$\begin{pmatrix} I_r & S & T \\ 0 & I_p & Q \\ 0 & 0 & 1 \end{pmatrix}$$

where I_r, I_p are the identity matrices of order $r\times r$ and $p\times p$ respectively, S is a $(r\times p)$-matrix, $T\in\mathbb{R}^r$ and $Q\in\mathbb{R}^p$.

We consider the immersion i: $H(1, 1) \to H(2, 2)$ given by

$$\begin{pmatrix} 1 & a_{12} & a_{13} \\ 0 & 1 & a_{23} \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & a_{12} & 0 & a_{13} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & a_{23} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

which is compatible with the group operation. There exists a left-invariant almost Hermitian structure J on $H(2, 2)$ [14] such that $H(1, 1)$ is a semi-invariant submanifold of $H(2, 2)$ with respect to a left-invariant unit vector field U parallel on $H(1, 1)$.

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Abstract

This paper is devoted to extend some results obtained about invariant submanifolds and complex hypersurfaces of Kaehlerian manifolds to semi-invariant submanifolds, giving many interesting examples of submanifolds of this kind.

The definition of a semi-invariant submanifold of an almost complex manifold is given in 1. *In* 2 *some properties of semi-invariant submanifolds immersed in a Kaehlerian manifold are considered. The cases where the submanifold has trivial normal connection and where the ambient manifold has constant holomorphic sectional curvature are studied in* 3 *and* 4, *respectively. Also, in addition to the examples of each paragraph, further examples of semi-invariant submanifolds are given in* 5.

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