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On a theorem of Kiričenko relating to 3-symmetric spaces (**)

1 – In this note we consider the following theorem due to Kiričenko [3].

Theorem A. The class of Hermitian-homogeneous naturally reductive almost Hermitian manifolds coincides with the class of analytic nearly Kähler manifolds that are locally 3-symmetric spaces.

Our purpose is to show that the restriction of being naturally reductive can be removed so as to obtain a complete correspondence between Hermitian-homogeneous almost Hermitian manifolds and locally 3-symmetric spaces. More precisely

Theorem B. *Any pseudo-Riemannian locally 3-symmetric space is a Hermitian-homogeneous almost Hermitian manifold with respect to its canonical almost complex structure. Conversely, any pseudo-Riemannian Hermitian-homogeneous almost Hermitian manifold (M, g, J) is a locally 3-symmetric space with J as canonical almost complex structure.*

2 – We begin with some definitions and basic properties. Any pseudo-Riemannian manifold (M, g) is assumed to be smooth and finite dimensional. Write $\mathcal{T}^p_q$ for the algebra of smooth tensor fields on M with contravariant and covariant orders p and q respectively; in particular, write $\mathcal{T}^p_0 = \mathcal{T}^p$ and $\mathcal{T}^0_q = \mathcal{T}_q$.

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Let ∇ denote the Riemannian connection and R denote the Riemannian curvature tensor field on M, where the curvature operator R_{XY} is defined by

$$R_{XY} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$$

for all $X, Y \in \mathcal{T}^1$.

Now, any $S \in \mathcal{T}^1_1$ can be considered as a field of linear endomorphisms of tangent spaces of M. Then we say that any $T \in \mathcal{T}^p_q$ is *S-invariant* if for all $\omega_1, \dots, \omega_p \in \mathcal{T}_1$ and $X_1, \dots, X_q \in \mathcal{T}^1$

$$T(\omega_1 S, \dots, \omega_p S, X_1, \dots, X_q) = T(\omega_1, \dots, \omega_p, SX_1, \dots, SX_q)$$

where $(\omega S)(X) = \omega(SX)$ for $\omega \in \mathcal{T}_1$ and $X \in \mathcal{T}^1$. We call S a *symmetry tensor field* if I-S is non-singular and g is S-invariant. In particular, S is said to be *regular* if ∇S and $\nabla^2 S$ are S-invariant. Any *pseudo-Riemannian locally* 3*-symmetric space* can be defined either by local isometries or by the following tensor conditions which are more appropriate for our requirements (cf. [1] or [5]).

Def. Let S be a regular symmetry tensor field on (M, g) such that $S^3 = I$ and the tensor fields R and ∇R are S-invariant. Then the triple (M, g, S) is *a pseudo-Riemannian locally* 3*-symmetric space.*

We note that such (M, g, S) is almost Hermitian with respect to an almost complex structure J defined by

$$S = -\frac{1}{2}I + \frac{\sqrt{3}}{2}J. \tag{1}$$

J is called the *canonical* almost complex structure on (M, g, S) (cf., e.g., [2]).

On the other hand, let (M, g) be a locally homogeneous reductive pseudo-Riemannian manifold with homogeneous structure T (cf. [3] and [6]). Thus $T \in \mathcal{T}^1_2$ is given such that for all $X, Y, Z \in \mathcal{T}^1$

$$g(T(X, Y), Z) + g(Y, T(X, Z)) = 0 \tag{P1}$$

$$\tilde{\nabla} R = 0 \quad \text{(P2)} \qquad\qquad \tilde{\nabla} T = 0 \quad \text{(P3)}$$

where $\tilde{\nabla}$ is the connection defined by $\tilde{\nabla} = \nabla - T$. Then (M, g) is said to be

Hermitian-homogeneous [3] if it is almost Hermitian with respect to an almost complex structure J satisfying

(P4) $\bar{\nabla} J = 0$ (P5) $T(JX,\ Y) = T(X,\ JY) = -JT(X,\ Y)$.

We write $T(X,\ Y) = T_X Y$ when T_X is considered as a derivation on the algebra of tensor fields on M.

3 – Now we are ready to prove our result.

Proof of Theorem B. Suppose $(M,\ g,\ S)$ is given with canonical almost complex structure J as above. Then it is well-known [**1**] that the tensor field T given by

$$T(X,\ Y) = (\nabla_{(I-S)^{-1}X} S)\, S^{-1} Y \tag{2}$$

for all $X,\ Y \in \mathcal{T}^1$, is a homogeneous structure on M. Since ∇S is S-invariant, then so is ∇J and, as an easy consequence, we obtain

$$(\nabla_{JX} J)\, Y + J(\nabla_X J)\, Y = 0 \tag{3}$$

for all $X,\ Y \in \mathcal{T}^1$. Then (2) simplifies to give

$$T(X,\ Y) = \frac{1}{2} J(\nabla_X J)\, Y\ . \tag{4}$$

From (4) we have

$$(T_X J)\, Y = (\nabla_X J)\, Y \tag{5}$$

which is equivalent to

$$\bar{\nabla} J = 0$$

for the canonical connection $\bar{\nabla} = \nabla - T$. Also, we note that (3) and (5) can be written, respectively, as

$$T(JX,\ Y)+JT(X,\ Y)=0 \quad \text{and} \quad T(X,\ JY)+JT(X,\ Y)=0\ .$$

Thus $(M,\ g)$ is Hermitian-homogeneous.

Conversely, suppose given a pseudo-Riemannian Hermitian-homogeneous almost Hermitian manifold $(M,\ g,\ J)$ with homogeneous structure T. Define S by (1). Then $I-S$ is non-singular and g is S-invariant. Also, T is S-invariant from (P5) and $\tilde{\nabla}S=0$ from (P4). It follows immediately from the relation $\nabla=\tilde{\nabla}+T$ that ∇S is S-invariant. Next, we use (P3) to obtain

$$\begin{aligned}\nabla^2_{XY}S &= \nabla_X(\nabla_Y S)-\nabla_{\nabla_X Y}S=(\tilde{\nabla}_X+T_X)(\tilde{\nabla}_Y+T_Y)S-T_{\nabla_X Y}S\\ &= T_X(T_Y S)+T_{\tilde{\nabla}_X Y}S-T_{\nabla_X Y}S=T_X(T_Y S)-T_{T_X Y}S\end{aligned}$$

which is S-invariant. Hence S is a regular symmetry tensor field for which $S^3=I$. Further, we prove that R is S-invariant. Write $\tilde{R}$ for the curvature tensor of $\tilde{\nabla}$ and define $P\in\mathcal{T}^1_3$ by

$$P(X,\ Y,\ Z)=P_{XY}Z=T_{T_XY-T_YX}Z-T_XT_YZ+T_YT_XZ$$

for all $X,\ Y,\ Z\in\mathcal{T}^1$. Since $\tilde{\nabla}T=0$, it follows that

$$\tilde{R}_{XY}Z=R_{XY}Z-P_{XY}Z$$

(cf., e.g., [**6**], p. 15). By taking inner products with W we see that this equation has the covariant form

$$\tilde{R}_{XYZW}=R_{XYZW}+g(T_XZ,\ T_YW)-g(T_YZ,\ T_XW)-g(T_{T_XY-T_YX}Z,\ W)\ .$$

Since T is S-invariant, this implies

$$\tilde{R}_{SXSYSZSW}-\tilde{R}_{XYZW}=R_{SXSYSZSW}-R_{XYZW}\ . \tag{6}$$

Next, define

$$A_{XYZW}=R_{SXSYSZSW}-R_{XYZW}\ .$$

Then A satisfies the identities of a Riemannian curvature tensor. Also, since $\bar{\nabla}J=0$, we have from (6)

$$A_{XYJZJW}=A_{XYZW}\,. \tag{7}$$

Furthermore, as an easy consequence of (1),

$$A_{XJXXJX}=0\,. \tag{8}$$

Then (7) and (8) imply $A=0$ [4]. Thus R is S-invariant. Finally, $\bar{\nabla}R=0$ implies

$$\nabla_X R=T_X R$$

for all $X\in\mathcal{T}^1$. Since T and R are S-invariant, it follows that ∇R is S-invariant. This proves that $(M,\ g,\ J)$ is a locally 3-symmetric space with J as canonical almost complex structure. The proof is now complete.

Remark. Kiričenko defines a locally homogeneous reductive pseudo-Riemannian manifold to be *naturally reductive* if its associated Lie algebra has the usual naturally reductive property. But this is equivalent to the canonical connection $\bar{\nabla}$ and Riemannian connection ∇ having the same geodesics [4]. In turn, this is equivalent to the homogeneous structure T satisfying

$$T_X X=0 \tag{9}$$

for all $X\in\mathcal{T}^1$, since $T_X X=\nabla_X X-\bar{\nabla}_X X=0$. Clearly, (9) reduces to the nearly Kähler property $(\nabla_X J)X=0$ when T satisfies (4). Then Kiričenko's result (Theorem A) follows immediately.

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Abstrait

On donne une charactérisation locale des variétés 3-*symétriques à l'aide des structures homogènes.*

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