Riv. Mat. Univ. Parma (4) **13** (1987), 401-405

FRANCIS J. SULLIVAN (*)

A note on the q-gamma functions (**)

Introduction

Recently Laugwitz and Rodewald [4] have given an elegant simple characterization of the classical gamma function. The most celebrated characterization of $\Gamma(x)$ is undoubtedly that of Bohr-Mollerup [2]: $\Gamma(x)$ is uniquely characterized by the three conditions:

(a) $\Gamma(1)=1$ (b) $\Gamma(x+1)=x\Gamma(x)$ (c) $\log(\Gamma(x))$ is a convex function of x for $x>0$.

Laugwitz and Rodewald prove the following

Theorem. There exists a unique function Γ, $\Gamma(x)>0$ for $x\geq 1$, such that $L(x)=\log\Gamma(x+1)$ has for $x\geq 0$, the following properties:

(i) $L(0)=0$ (ii) $L(x+1)=\log(x+1)+L(x)$ (iii) $L(n+x) = L(n)+x\log(n+1)+r_n(x)$ where $\lim_{n\to\infty} r_n(x)=0$.

The motivation for this result dates back to Euler. It lies in the fact that for a fixed very large integer n and all positive integers m the sequence of factorials $(n+m)!$ behaves very much like a geometric sequence. In other words

$$L(n+m)=L(n)+m\log(n+1)+\sum_{k=1}^{m}\log(1+\frac{k-1}{n+1})\simeq L(n)+m\log(n+1)\ .$$

(*) Indirizzo: Dipartimento di Matematica, Via Belzoni 7, I-35100 Padova.
(**) Ricevuto: 7-X-1987.

Thus for integral values of m we see that $L(n+m)$ is approximately a linear function of m, and it is natural to follow Euler in supposing that linearity extends to all real values of x in the place of m.

It is well-known that there is a close similarity between the behavior of the gamma function and its q-analogues, defined by

$$\Gamma_q(x) = \frac{(q;\ q)_\infty (1-q)^{1-x}}{(q;\ q^x)_\infty} \quad \text{for } 0<q<1 \quad \text{and } x>0\ .$$

Here the symbol $(q;\ a)_\infty$ is defined by $(q;\ a)_\infty = \prod_{k=0}^{\infty} (1-aq^k)$ for arbitrary (even complex) a and q with $|q|<1$.

The latter functions, first introduced by Thomae [7], are interesting not only as a rather natural generalization of $\Gamma(x)$, but also for their connections with various other classical problems and functions, e.g., problems in the theory of partitions, moment problems, and Jacobi theta functions. They have recently been the subject of renewed interest, e.g., $[1]_{1,2}$, [3], [5], [6].

It is natural to inquire whether or not there is also a q-analogue to Laugwitz and Rodewald's «eulerian» characterization of $\Gamma(x)$. The goal of this brief note is to show that such is indeed the case.

1 – In addition to the symbol $(q;\ a)_\infty$ defined above we use the symbol $[x]$ to denote the «q-basic number» defined by $[x] = \frac{q^x-1}{q-1}$. The analogue of Laugwitz and Rodewald's result may then be stated as

Theorem. *For each q with $0<q<1$ there exists a unique function $\Gamma_q(x)$, with $\Gamma_q(x)>0$ for $x\geq 1$ such that $L_q(x) = \log(\Gamma_q(x+1)$ has, for $x\geq 0$ the following properties:*

$$\text{(i)}\quad L_q(0)=0 \qquad \text{(ii)}\quad L_q(x+1) = \log([x+1]) + L_q(x)$$

$$\text{(iii)}\quad L_q(n+x) = L_q(n) + x\log([n+1]) + r_{n,q}(x) \qquad \text{with}$$

$$\lim_{n\to\infty} r_{n,q}(x) = 0\ . \tag{1}$$

Proof. From (i) and (ii) we obtain

$$L_q(n) = \sum_{k=1}^{n} \log([k]) \tag{2}$$

and for a positive integer n and $x \geq 0$

$$(3) \qquad L_q(x+n) = L_q(x) + \sum_{k=1}^{n} \log([x+k]) .$$

It follows from equations (3), (iii) and (2) that

$$(4) \qquad L_q(x) = L_q(n) - \sum_{k=1}^{n} \log([x+k]) + x\log([n+1]) + r_{n,q}(x)$$

$$(5) \qquad L_q(x) = \sum_{k=1}^{n} \{\log[k] - \log([x+k]) + x(\log([k+1]) - \log([k]))\} + r_{n,q}(x)$$

$$(6) \qquad L_q(x) = \sum_{k=1}^{n} \{(1-x)\log([k]) + x\log([k+1]) - \log([x+k])\} + r_{n,q}(x) .$$

If we now define, in strict analogy with the classical Euler constant γ, the q-Euler constant γ_q by setting $\gamma_q = \lim_{n\to\infty} \gamma_{q,n}$ with

$$(7) \qquad \gamma_{q,n} = 1 + \frac{q}{1+q} + \frac{q^2}{1+q+q^2} + \ldots + \frac{q^{n-1}}{1+q+\ldots+q^{n-1}} - \log([n+1])$$

we then obtain

$$(8) \qquad L_q(x) = -\gamma_{q,n}x + \sum_{k=1}^{n} \{\frac{xq^{k-1}}{[k]} - \log([x+k]) + \log([k])\} + r_{n,q}(x)$$

that is

$$(9) \qquad L_q(x) = -\gamma_{q,n}x + \sum_{k=1}^{n} (\frac{xq^{k-1}}{[k]} - \log(\frac{[x+k]}{[k]})) + r_{n,q}(x) .$$

The series in (9) clearly converges for $x > 0$ since

$$\log(\frac{[x+k]}{[k]}) = \log(\frac{1-q^{x+k}}{1-q^k}) = \log(1-q^{x+k}) - \log(1-q^k) < 4q^k$$

for large enough k (i.e. for $q^k < \frac{1}{2}$) and $\frac{xq^{k-1}}{[k]} < xq^{k-1}$.

It follows that the right-hand side of (9) and so also the right-hand sides of (8), (6), (5) and (4) converge to a uniquely determined real-valued function $L(x)$

defined for $x \geq 0$ by

$$L_q(x) = -\gamma_q x + \sum_{k=1}^{\infty} \left(\frac{xq^{k-1}}{[k]} - \log\left(\frac{[x+k]}{[k]}\right)\right). \tag{10}$$

There remains only to prove that $L_q(x)$ has properties (i), (ii) and (iii).

Property (i) is obvious form the definition (10) of $L_q(x)$. From equations (5) and (1) we find that

$$L_q(x+1) - L_q(x) = \sum_{k=1}^{n} \{\log([x+k]) - \log([x+1+k]) + \log([k+1]) - \log([k])\}$$
$$+ r_{n,q}(x+1) - r_{n,q}(x)$$
$$= \log([x+1]) - \log([x+n+1]) + \log[n+1] + r_{n,q}(x+1) - r_{n,q}(x)$$
$$= \log([x+1]) - \log([x+n+1]/[n+1]) + r_{n,q}(x+1) - r_{n,q}(x)$$

and, happily, we do get property (ii) as $n \to \infty$. This is the case since $[x+n+1]/[n+1] = 1 - q^{x+n+1}/1 - q^{n+1}$, so $\log([x+n+1]/[n+1])$ is dominated by $2(q^{n+1+x} + q^{n+1})$ for sufficiently large n. Equation (3) follows from (ii), and (4) is equivalent to (9). Property (iii) then follows from (3) and (4).

It is perhaps interesting to observe that, as might be expected, the modification of the functional equation of $\Gamma(x)$ leads to a different «slope at ∞», that is, the coefficient of x in (iii) changes from $\log(n+1)$ to $\log([n+1])$. Of course, as $q \to 1^-$ all our equations tend to those of Laugwitz and Rodewald.

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Sommario

Si dimostra un risultato analogo a quello ottenuto da Laugwitz e Rodewald nella loro nota A simple characterization of the Gamma function, Amer. Math. Monthly 6, 1987. *Tale risultato si ispira all'intuizione di Eulero per caratterizzare la funzione Gamma in modo semplice.*

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