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One theorem on asymptotic behaviour of solutions of a certain system of quasilinear differential equations not solved with respect to derivatives (**)

1 - Introduction

In this paper we consider two systems of singular ordinary differential equations

$$g_i(x)\,y_i' = a_i(x)\,y_i + \omega_i(x) + f_i(x,\ y_1,\ \dots,\ y_n,\ y_1',\ \dots,\ y_n') \qquad (i=1,\ \dots,\ n) \tag{1}$$

$$g_i(x)\,z_i' = a_i(x)\,z_i + \omega_i(x) \qquad (i=1,\ \dots,\ n)\ . \tag{2}$$

The following problem for (1), (2) is posed: If $z(x)=(z_1(x),\ \dots,\ z_n(x))^T$ is a solution of (2), is there a solution $y(x)=(y_1(x),\ \dots,\ y_n(x))^T$ of (1) such that for $x\to 0^+$ $y_i(x)\sim z_i(x)\sim\varphi_i(x)$ $(i=1,\ \dots,\ n)$ where $\varphi(x)=(\varphi_1(x),\ \dots,\ \varphi_n(x))^T$ is the asymptotic representation of solution $z(x)$ deduced from its integral from?

The point (0, 0) may be singular point for these systems. The functions $g_i(x)$, $a_i(x)$, $\omega_i(x)$, $f_i(x,\ y_1,\ \dots,\ y_n,\ y_1',\ \dots,\ y_n')$ satisfy some mentioned further conditions, but it is possible that $g_i(0^+)=0$, $a_i(0^+)=\omega_i(0^+)=\infty$ and $f_i(0^+,\ y_1,\ \dots,\ y_n,\ y_1',\ \dots,\ y_n')=\infty$ if $y_j,\ y_j'=\text{const}$ $(i=1,\ \dots,\ n;\ j=1,\ \dots,\ n)$.

Some related problems for systems solved with respect to derivatives were studied in the case where $\omega_i\equiv 0$ $(i=1,\ \dots,\ n)$, for example in papers [2], [3] and in the case where $f_i\equiv 0$ $(i=1,\ \dots,\ n)$ in papers [1] and [4].

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2 - Preliminary lemma

We consider the scalar equation

$$g(x)\,y' = a(x)\,y + \omega(x) \tag{3}$$

and give the asymptotic formula of its particular solution if $x \to 0^+$. We will suppose that there are such functions $f(x)$ and $\omega^1(x)$ that $(g(x),\ a(x),\ \omega(x),\ f(x),\ \omega^1(x)) \in \Omega$ that is if the following hypotheses (Q_1)-(Q_4) are satisfied:

(Q_1) $g(x) \in C^1(0,\ x_0],\ 0 < x_0 = \text{const},\ 0 < g(x)$ on $(0,\ x_0],\ a(x) \in C(0,\ x_0]$; $\omega^1(x) \in C^1(0,\ x_0],\ \omega^1(x) \neq 0,\ |\omega(x) - \omega^1(x)| < 1$ on $(0,\ x_0]$; $\lim_{x\to 0^+} \omega(x)\,[\omega^1(x)]^{-1} = 1$.

(Q_2) $f(x) \in C^2(0,\ x_0],\ f(x) \neq 0,\ g(x)f'(x) \neq a(x)f(x)$ on $(0,\ x_0]$.

(Q_3) $\lim_{x\to 0^+} \mathscr{L}_1(x,\ x_0) = A$ where $A = 0$ or $A = \infty$

$$\mathscr{L}_1(x,\ x_0) \equiv \mathscr{L}_0(x,\ f(x)) \exp \int_x^t \frac{a(s)\,\mathrm{d}s}{g(s)}$$

$$\mathscr{L}_0(x,\ f(x)) \equiv \omega^1(x) f(x)\,[g(x)f'(x) - a(x)f(x)]^{-1}\,.$$

(Q_4) $\lim_{x\to 0^+} \mathscr{L}_2(x) = 0$ where $\mathscr{L}_2(x) \equiv [\mathscr{L}_0(x,\ f(x))\,(f(x))^{-1}]'\,f(x)\,g(x)\,(\omega^1(x))^{-1}$.

Lemma. *Suppose that* $(g(x),\ a(x),\ \omega(x),\ f(x),\ \omega^1(x)) \in \Omega$. *Then there is such particular solutions* $y(x)$ *of equation* (3) *that* $y(x) \sim \mathscr{L}_0(x,\ f(x))$ *as* $x \to 0^+$.

Proof. We choose the particular solution $y(x)$ of (3) in the integral form as

$$y(x) = \int_{x_1}^{x} \left(\frac{\omega(t)}{g(t)} \exp \int_t^x \frac{a(s)\,\mathrm{d}s}{g(s)}\right) \mathrm{d}t \tag{4}$$

where $x_1 = 0^+$ is put if $A = 0$ and $x_1 = x_0$ if $A = \infty$. From (4) we deduce that

$$y(x) = \int_{x_1}^{x} I_1(t)\,\mathrm{d}t + \int_{x_1}^{x} b(t)\,I_1(t)\,\mathrm{d}t$$

where $I_1(t) \equiv \dfrac{\mathscr{L}_0(t,\ f(t))}{f(t)} [f(t) \exp \int_t^x \dfrac{a(s)\,\mathrm{d}s}{g(s)}]_t'$ and $b(x) \equiv \omega(x) - \omega^1(x)$.

Integrating by parts we obtain

$$\int_{x_1}^{x} I_1(t)\,\mathrm{d}t \equiv \mathscr{L}_0(x,\ f(x)) - \mathscr{L}_1(x_1,\ x) - I_2(x)$$

where $I_2(x) \equiv \int_{x_1}^{x} (\mathscr{L}_2(t) \dfrac{\omega^1(t)}{g(t)} \exp \int_t^x \dfrac{a(s)\,\mathrm{d}s}{g(s)})\,\mathrm{d}t.$

By means of condition (Q_3) we may verify that

$$\lim_{x\to 0^+} \frac{\mathscr{L}_1(x_1,\ x)}{\mathscr{L}_0(x,\ f(x))} = \lim_{x\to 0^+} \frac{\mathscr{L}_1(x_1,\ x_0)}{\mathscr{L}_1(x,\ x_0)} = 0\ .$$

Let us show that

$$\lim_{x\to 0^+} I_3(x) = 0 \qquad \text{where} \qquad I_3(x) \equiv I_2(x)\,[\mathscr{L}_0(x,\ f(x))]^{-1}\ .$$

At first we suppose that $x_1 = 0^+$ and choose such monotically decreasing sequence of positive numbers $\{x_n\} \to 0$ that for $x \in [0,\ x_n]$ the inequalities $|\mathscr{L}_2(x)| \leqslant \dfrac{1}{n}$, $|\dfrac{\mathscr{L}_1(x_1,\ x)}{\mathscr{L}_0(x,\ f(x))}| \leqslant \dfrac{1}{n}$ $(n = 2,\ 3,\ \ldots)$ hold. Then it is easy to verify that

$$|I_3(x)| \leqslant \frac{1}{n} |(\mathscr{L}_0(x,\ f(x)))^{-1} \int_{0^+}^{x} I_1(t)\,\mathrm{d}t| \leqslant \frac{1}{n}(1 + \frac{1}{n} + |I_3(x)|)$$

and consequently, $|I_3(x)| \leqslant \dfrac{n+1}{n(n-1)}$. Therefore $I_3(0^+) = 0$. If $x_1 = x_0$ then we may rewrite $I_3(x)$ in the form

$$I_3(x) \equiv \frac{1}{\mathscr{L}_1(x,\ x_0)} \int_{x_0}^{x} (\mathscr{L}_2(t) \frac{\omega_1(t)}{g(t)} \exp \int_t^{x_0} \frac{a(s)\,\mathrm{d}s}{g(s)})\,\mathrm{d}t\ .$$

If the numerator of this fraction converges at $x \to 0^+$ then in view of pressumption (Q_3) we conclude that $I_3(0^+) = 0$. In the opposite case, using L'Hospital's rule, we obtain

$$\lim_{x\to 0^+} I_3(x) = \lim_{x\to 0^+} \frac{\mathscr{L}_2(x)}{\mathscr{L}_2(x) + 1} = 0\ .$$

Similarly we can prove that

$$\lim_{x\to 0^+}\Big[\int_{x_1}^{x} I_1(t)\,dt\Big]^{-1}\Big[\int_{x_1}^{x} b(t)\,I_1(t)\,dt\Big]=0\ .$$

The lemma is proved.

3 - Result

Let us denote $Y^{(j)}=(Y_1^{(j)},\ \dots,\ Y_n^{(j)})$ $(j=0,\ 1)$;

$$\eta_i(x,\ Y_i)\equiv z_i(x)+Y_i\exp\int_{x_0}^{x}\frac{a_i(s)\,ds}{g_i(s)}$$

where: $i=1,\ \dots,\ n$; $z_i(x)$ are the coordinates of some particular solution $z(x)=z_1(x),\ \dots,\ z_n(x))^T$ of system (2); $\xi=(\xi_1,\ \dots,\ \xi_n)^T$ with

$$\xi_i=\xi_i(x,\ Y,\ Y')$$

$$=\exp\int_{x_0}^{x}\frac{a_i(s)\,ds}{g_i(s)}\cdot f_i(x,\ \eta_1(x,\ Y_1),\ \dots,\ \eta_n(x,\ Y_n),\ \eta_1'(x,\ Y_1)\ \dots,\ \eta_n'(x,\ Y_n))\ .$$

Further we will consider the system

$$Y'=\xi(x,\ Y,\ Y') \tag{5}$$

in the region $D[(x,\ Y,\ Y')\colon 0<x\leq x_0,\ \|Y\|\leq\delta_0(x),\ \|Y'\|\leq\delta_1(x)]$, where $\|\cdot\|$ is the Euclidean norm, $0<\delta_i(x)$ on $(0,\ x_0]$, $\delta_i(x)\in C(0,\ x_0]$, $(i=0,\ 1)$, $\delta_0(0^+)=0$, $\lim_{x\to 0^+}\delta_0(x)\exp\int_{x_0}^{x}\frac{a_j(s)\,ds}{g_j(s)}=0$ $(j=1,\ \dots,\ n)$, $\int_{0^+}^{x}\delta_1(t)\,dt\leq\delta_0(x)$ on $(0,\ x_0]$.

Theorem. *Suppose that there are the functions $f_i(x)$, $\omega_i^1(x)$ $(i=1,\ \dots,\ n)$ such that the following assumptions $(Q_1)'$ ... $(Q_4)'$ for $(i=1,\ \dots,\ n)$ hold:*

$(Q_1)'$ $g_i(x)\in C^1(0,\ x_0]$, $0<g_i(x)$ *on* $(0,\ x_0]$; $a_i(x),\ \omega_i(x)\in C(0,\ x_0]$, $\omega_i^1(x)\,C^1(0,\ x_0]$, $\omega_i^1(x)\neq 0$, $|\omega_i(x)-\omega_i^1(x)|<1$ *on* $(0,\ x_0]$; $\lim_{x\to 0^+}\omega_i(x)[\omega_i^1(x)]^{-1}=1$.

$(Q_2)'$ $f_i(x)\in C^2(0,\ x_0]$, $f_i(x)\neq 0$; $g_i(x)f_i'(x)\neq a_i(x)f_i(x)$ *on* $(0,\ x_0]$.

$(Q_3)'$ $\quad \lim_{x \to 0^+} \mathscr{L}_{1i}(x, x_0) = A_i$ *where* $A_i = 0$ *or* $A_i = \infty$,

$$\mathscr{L}_{1i}(x, x_0) \equiv \mathscr{L}_{0i}(x, f_i(x)) \exp \int_x^t \frac{a_i(s)\,\mathrm{d}s}{g_i(s)},$$

$$\mathscr{L}_{0i}(x, f_i(x)) \equiv \omega_i^1(x) f_i(x) [g_i(x) f_i'(x) - a_i(x) f_i(x)]^{-1}.$$

$(Q_4)'$ $\quad \lim_{x \to 0^+} \mathscr{L}_{2i}(x) = 0$ where $\mathscr{L}_{2i}(x) \equiv [\mathscr{L}_{0i}(x, f_i(x)) (f_i(x))^{-1}]' \dfrac{f_i(x)\, g_i(x)}{\omega_i^1(x)}$.

Then there is such particular solution $z(x) = (z_1(x), \ldots, z_n(x))^T$ *of system* (2) *that for* $x \to 0^+$ *the following representations*

$$z_i(x) \sim \varphi_i(x) \tag{6}$$

where $\varphi_i(x) \equiv \mathscr{L}_{0i}(x, f_i(x))$ $(i = 1, \ldots, n)$, *hold.*

Proof. For each $i = 1, \ldots, n$ the inclusion $(g_i(x), a_i(x), \omega_i(x), f_i(x), \omega_i^1(x)) \in \Omega$ hold. Therefore for each equation of system (2) the conditions of Lemma are valid. From conclusion of this lemma the conclusion of the theorem follows immediately.

Main theorem. *Let the pressumptions* $(Q_1)' \ldots (Q_4)'$ *of theorem hold. Let, moreover, in region* D *the following assumptions be satisfied:*

$$\xi(x, Y, Y') \in C \qquad \|\xi(x, Y, Y')\| \leq \delta_1(x)$$

$$\|\xi(x, \bar{Y}, \bar{Y}') - \xi(x, \bar{\bar{Y}}, \bar{\bar{Y}}')\| \leq M \|\bar{Y} - \bar{\bar{Y}}\| + N \|\bar{Y}' - \bar{\bar{Y}}'\|$$

where $0 \leq M$, $N = \text{const}$, $N < 1$ *and* $z(x)$ *is the particular solution of system* (2) *given by previous theorem and represented by formulas* (6).

Then there is such solution $y(x) = (y_1(x), \ldots, y_n(x))^T$ *of system* (1) *defined on* $(0, x_0]$ *that the following representations*

$$y_i(x) \sim z_i(x) \qquad (i = 1, \ldots, n)$$

as $x \to 0^+$ *hold.*

Proof. Let us put in system (1)

$$y_i = z_i(x) + Y_i \exp \int_{x_0}^{x} \frac{a_i(s)\,\mathrm{d}s}{g_i(s)} \tag{7}$$

where Y_i $(i=1, \ldots, n)$ are new variables and $z(x)=(z_1(x), \ldots, z_n(x))^T$ is the above-mentioned particular solution of system (2). Then Y_i $(i=1, \ldots, n)$ satisfy to the system (5). We prove that there is such solution $Y(x)$ of (5) that $(x, Y(x), Y'(x)) \in D$ on interval $(0, x_0]$. At first we prove that the system of implicit equations $w=\xi(x, Y, w)$, where $w=(w_1, \ldots, w_n)^T$, defines in the domain $D_1[0<x<x_0, \|Y\| \leqslant \delta_0(x)]$ unique solution $w=w(x, Y) \in C(D_1)$ such that there the inequality $\|w(x, Y)\| \leqslant \delta_1(x)$ holds. Let A be operator acting by formula $Aw=\xi(x, Y, w)$ where $(x, Y, w) \in D$. Then $Aw \in C(D_1)$, $\|Aw\| \leqslant \delta_1(x)$ and for w^i where $\|w^i\| \leqslant \delta_1(x)$ $(i=1, 2)$ and for metric $\rho(w^1, w^2)=\|w^1-w^2\|$ it holds: $\rho(Aw^1, Aw^2) \leqslant N\rho(w^1, w^2)$. Therefore A is the contracting operator and from this fact it follows that above formulated affirmation is true. Consequently the system (5) may be in the region D_1 rewritten in equivalent form

$$Y' = w(x, Y) \tag{8}$$

where $w(x, Y) \in C$, $\|w(x, Y)\| \leqslant \delta_1(x)$ and moreover $\|w(x, \bar{Y})-w(x, \bar{\bar{Y}})\| \leqslant M(1-N)^{-1}\|\bar{Y}-\bar{\bar{Y}}\|$. Further we will apply the norm $\|u(x)\|^0 = \sup_{t\in(0, x_2)} \|u(t)\|$ and the metric $\rho(u^1(x), u^2(x)) = \|u^1(x)-u^2(x)\|^0$ to the set of functions $U=\{u(x)\}$ such that for $u(x) \in U$: $(x, u(x)) \in D_1$ on $(0, x_2)$, where $x_2=\text{const}$, $0<x_2<\min\{M^{-1}(1-N), x_0\}$ and $u(x) \in C(0, x_2)$. Let B be operator acting by formula $Bu(t)=\int_{0^+}^{x} w(t, u(t))\,dt$ where $u(x) \in U$. The operator B maps the set of functions U into itself and, moreover, $\rho(Bu^1, Bu^2) \leqslant (1-N)^{-1}Mx\rho(u^1, u^2)$ if $u^1, u^2 \in U$.

Henceforward operator B is on interval $(0, x_2)$ the contracting operator and problem (8) has a unique solution $Y(x)$ such that $(x, Y(x)) \in D_1$ on $(0, x_2)$. In the case when $M^{-1}(1-N)<x_0$ holds we may continue this solution continuously on interval $(0, x_0]$ in accordance with classical existence and uniqueness theorems and in view of inequality $\|Y'(x)\| \leqslant \delta_1(x)$. Consequently on interval $(0, x_0]$ the inclusion $(x, Y(x)) \in D_1$ holds. Finally from the substitutions (7) we conclude that there is solution $y(x)=(y_1(x), \ldots, y_n(x))^T$ of system (1) such that

$$|y_i(x)-z_i(x)| = \left|Y_i(x) \exp \int_{x_0}^{x} \frac{a_i(s)\,ds}{g_i(s)}\right| \leqslant \delta_0(x) \exp \int_{x_0}^{x} \frac{a_i(s)\,ds}{g_i(s)}$$

$(i=1, \ldots, n)$. Right-hands of this inequalities converge to zero if $x \to 0^+$ and this fact concludes the proof.

References

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Zusammenfassung

In der vorgelegten Arbeit wird der asymptotische Charakter der Lösung des Systems von Differentialgleichungen $g_i(x)y_i' = a_i(x)y_i + \omega_i(x)(1 + f_i(x, y_1, \ldots, y_n, y_1', \ldots, y_n'))$ $(i = 1, \ldots, n)$ *in der Umgebung des singulären Punktes untersucht.*

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