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A unified presentation of certain classes of series of the Riemann zeta function (**)

1 - Introduction

An over two-century old theorem of Christian Goldbach (1690-1764), which was stated in a letter dated 1729 from Goldbach to Daniel Bernoulli (1700-1782), has recently been posed as the following

Problem (Shallit and Zikan [26]). Let $\mathcal{S}$ be the set of nontrivial integer kth powers, i.e.,

$$\mathcal{S} = \{n^k | n \geqq 2,\ k \geqq 2\} = \{4, 8, 9, 16, 25, 27, 32, 36, \dots\}. \tag{1.1}$$

Show that

$$\sum_{\omega \in \mathcal{S}} (\omega - 1)^{-1} = 1 \tag{1.2}$$

the sum being extended over all members ω of $\mathcal{S}$.

In terms of the Riemann zeta function (see Titchmarsh [30] and Ivić [13])

$$\zeta(s) = \begin{cases} \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1-2^{-s}} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} & \mathrm{Re}(s) > 1 \\ \frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} & \mathrm{Re}(s) > 0 \quad s \neq 1 \end{cases} \tag{1.3}$$

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the summation formula (1.2) becomes (cf. [26], p. 403)

$$\sum_{k=2}^{\infty}\{\zeta(k)-1\}=1. \tag{1.4}$$

More interestingly, since (for $k\geqq 2$)

$$1<\zeta(k)\leqq\zeta(2)=\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}<2$$

giving us $0<\zeta(k)-1<1$, $k\geqq 2$ so that

$$\zeta(k)-1=f(\zeta(k)) \qquad k\geqq 2 \tag{1.5}$$

where $f(x)=x-[x]$ denotes the fractional part of the real number x, (1.4) can be rewritten in the elegant form

$$\sum_{k=2}^{\infty}f(\zeta(k))=1. \tag{1.6}$$

As a matter of fact, it is not difficult to show also that

$$\sum_{k=2}^{\infty}(-1)^k f(\zeta(k))=\frac{1}{2} \tag{1.7}$$

$$\sum_{k=1}^{\infty}f(\zeta(2k))=\frac{3}{4} \qquad \sum_{k=1}^{\infty}f(\zeta(2k+1))=\frac{1}{4}. \tag{1.8}$$

Formula (1.6), and hence also (1.2) and (1.4), and its interesting variations (1.7) and (1.8) are, of course, equivalent to various (known or easily derivable) sums of double series (see, for example, Boole [4], p. 105, Exercise 10; Stieltjes [28], p. 300; Johnson [14], p. 479; Bromwich [5], p. 526, Example 6; Jordan [15], p. 340; Chrystal [7], p. 422, Exercise 18; Melzak [21], p. 88; Klambauer [16], p. 120, Exercise 38 and Hansen [12], p. 355). The object of the present paper is to address several related problems involving sums of series of $\zeta(s)$ and of the generalized (Hurwitz's) zeta function $\zeta(s, a)$ defined usually by (cf. [8], p. 24, Equation 1.10(1))

$$\zeta(s, a)=\sum_{n=0}^{\infty}\frac{1}{(n+a)^s} \qquad \mathrm{Re}(s)>1 \quad a\neq 0, -1, -2, \ldots \tag{1.9}$$

so that, obviously,

$$\zeta(s,\ 1) = \zeta(s) \qquad \zeta(s,\ \frac{1}{2}) = (2^s - 1)\,\zeta(s) \tag{1.10}$$

$$\frac{\partial}{\partial a}\{\zeta(s,\ a)\} = -\,s\zeta(s+1,\ a) \tag{1.11}$$

$$\zeta(s,\ a+N) = \zeta(s,\ a) - \sum_{n=0}^{N-1} \frac{1}{(n+a)^s} \qquad N = 1,\ 2,\ 3,\ \dots. \tag{1.12}$$

It should be remarked in passing that both $\zeta(s)$ and $\zeta(s,\ a)$ are meromorphic functions everywhere in the complex s-plane except for a simple pole at $s = 1$ (with residue 1), and that

$$\zeta(0) = -\frac{1}{2} \qquad \zeta(0,\ a) = \frac{1}{2} - a. \tag{1.13}$$

2 - Generalizations of the sums (1.6) and (1.7)

In the usual notations for binomial coefficients, let

$$\binom{\lambda}{0} = 1 \qquad \binom{\lambda}{n} = \frac{\lambda(\lambda-1)(\lambda-2)\dots(\lambda-n+1)}{n!} \qquad n = 1,\ 2,\ 3,\ \dots \tag{2.1}$$

for an arbitrary (real or complex) parameter λ. Making use of the binomial expansion

$$\sum_{k=0}^{\infty} \binom{\lambda+k-1}{k} t^k = (1-t)^{-\lambda} \qquad |t| < 1 \tag{2.2}$$

it is easily seen from the definitions (1.3) and (1.9) that

$$\sum_{k=0}^{\infty} \binom{\lambda+k-1}{k} \{\zeta(\lambda+k) - 1\}\, t^k = \zeta(\lambda,\ 2-t) \qquad |t| < 2 \tag{2.3}$$

or, equivalently, that (cf. Ramanujan [23], p. 78, Equation (15); Apostol [2], p.

240, Equation (7))

$$\sum_{k=0}^{\infty} \binom{\lambda+k-1}{k} \zeta(\lambda+k)\, t^k = \zeta(\lambda,\ 1-t) \qquad |t|<1. \tag{2.4}$$

For fixed $\lambda \neq 1$, the series in (2.3) and (2.4) converge absolutely for $|t|<2$ and $|t|<1$, respectively. Thus, by the principle of analytic continuation, formulas (2.3) and (2.4) are valid for all values of $\lambda \neq 1$.

Formula (2.3) provides a unification (and generalization) of (1.6) and (1.7), and indeed also of a fairly large number of other summation formulas scattered in the literature. For example, in view of the relationships (1.10) and (1.12), (2.3) with $t=1$ gives us (cf. [**12**], p. 356, Equation (54.4.1))

$$\sum_{k=1}^{\infty} \binom{\lambda+k-1}{k} \{\zeta(\lambda+k)-1\} = 1 \tag{2.5}$$

which generalizes (1.6), and a special case of (2.3) when $t=-1$ yields (cf. [**12**], p. 356, Equation (54.4.2))

$$\sum_{k=1}^{\infty} (-1)^{k-1} \binom{\lambda+k-1}{k} \{\zeta(\lambda+k)-1\} = 2^{-\lambda} \tag{2.6}$$

which generalizes (1.7).

Several additional consequences of the general summation formulas (2.3) and (2.4) are worthy of note. First of all, replace the summation index k in (2.3) by $k+1$, and set $\lambda = s-1$, so that

$$\sum_{k=0}^{\infty} \binom{s+k-1}{k+1} \{\zeta(s+k)-1\}\, t^{k+1} = \zeta(s-1,\ 2-t) - \zeta(s-1) + 1 \qquad |t|<2 \tag{2.7}$$

which, for $t=1$, reduces immediately to the following alternative form of (2.5)

$$\sum_{k=0}^{\infty} \binom{s+k-1}{k+1} \{\zeta(s+k)-1\} = 1. \tag{2.8}$$

Now it follows from the definition (2.1) that

$$\binom{s+k-1}{k+1} = \frac{(s-1)(s)_k}{(k+1)!} \qquad k \geqq 0 \tag{2.9}$$

where, for convenience,

$$(s)_0 = 1 \qquad (s)_k = s(s+1)(s+2)\dots(s+k-1) \qquad k = 1,\ 2,\ 3,\ \dots. \tag{2.10}$$

Thus the formula (2.8) can be rewritten in the well-known form (cf. Landau [18], p. 274, Equation (3); Titchmarsh [30], p. 33, Equation (2.14.1))

$$\zeta(s) = 1 + \frac{1}{s-1} - \sum_{k=1}^{\infty} \frac{(s)_k}{(k+1)!} \{\zeta(s+k) - 1\} \tag{2.11}$$

which is usually attributed to Edmund (Georg Hermann) Landau (1877-1938).

For $t = -1$, (2.7) readily yields

$$\zeta(s) = 1 + \frac{1}{2^{s-1}} \frac{1}{s-1} + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{(s)_k}{(k+1)!} \{\zeta(s+k) - 1\} \tag{2.12}$$

which provides an interesting (presumably new) companion of Landau's formula (2.11).

Setting $t = 1/2$ in (2.7), and making use of (1.12) with $a = 1/2$ and $N = 1$, we obtain another series representation for $\zeta(s)$

$$\zeta(s) = \frac{2^s - 1}{2^s - 2} + \frac{1}{2^s - 2} \sum_{k=1}^{\infty} \frac{(s)_k}{k!\, 2^k} \{\zeta(s+k) - 1\} \tag{2.13}$$

which is believed to be new.

In their special cases when $s = 2$, (2.11) and (2.12) reduce simply to the summation formulas (1.6) and (1.7), respectively, while (2.13) similarly yields the elegant sum

$$\sum_{k=2}^{\infty} \frac{k-1}{2^k} \{\zeta(k) - 1\} = \frac{\pi^2}{8} - 1. \tag{2.14}$$

Next we turn to the summation formula (2.4) which (for $\lambda = s - 1$ and with k replaced by $k + 1$) assumes the form

$$\sum_{k=0}^{\infty} \binom{s+k-1}{k+1} \zeta(s+k)\, t^{k+1} = \zeta(s-1,\ 1-t) - \zeta(s-1) \qquad |t| < 1. \tag{2.15}$$

In view of the identities in (1.10) and (2.9), a special case of (2.15) when $t = 1/2$ readily yields the familiar result [1]

$$(1 - 2^{1-s})\,\zeta(s) = \sum_{k=1}^{\infty} \frac{(s)_k}{k!} \frac{\zeta(s+k)}{2^{s+k}} \tag{2.16}$$

which is attributed to Ramaswami (cf. [24], p. 166 and [30], p. 33, Equation (2.14.2)). Furthermore, in its special case when $t = -1/2$, (2.15) gives us the following companion of (2.16)

$$(1 - 2^{1-s})\,\zeta(s) = 1 - \sum_{k=1}^{\infty} (-1)^{k-1} \frac{(s)_k}{k!} \frac{\zeta(s+k)}{2^{s+k}} \tag{2.17}$$

which was also given by Ramaswami ([24], p. 166).

Formulas (2.11) and (2.16) were rederived, using Eulerian integrals for Γ-functions, by Menon [22].

In case we add (2.4) to itself (with t replaced by $-t$), we obtain the summation formula (cf. [12], p. 357, Equation (54.6.3))

$$\sum_{k=0}^{\infty} \binom{\lambda + 2k - 1}{2k} \zeta(\lambda + 2k)\, t^{2k} = \frac{1}{2} \{\zeta(\lambda,\ 1 - t) + \zeta(\lambda,\ 1 + t)\} \qquad |t| < 1 \tag{2.18}$$

while a similar subtraction yields

$$\sum_{k=0}^{\infty} \binom{\lambda + 2k}{2k + 1} \zeta(\lambda + 2k + 1)\, t^{2k+1} = \frac{1}{2} \{\zeta(\lambda,\ 1 - t) - \zeta(\lambda,\ 1 + t)\} \qquad |t| < 1. \tag{2.19}$$

Various interesting special cases of (2.18) and (2.19) are given in the literature. In particular, the special cases of (2.18) when $t = 1/2$, $t = 1/3$, and $t = 1/6$ were considered by Ramaswami ([24], p. 167, Equations (1), (3) and (4)) who also gave a special case of (2.19) when $t = 1/2$ ([24], p. 167, Equation (2)), and by Apostol [2] who proved various generalizations of Ramaswami's results.

By assigning suitable numerical values to the variable s in some of the aforementioned special cases of (2.18) and (2.19), Ramaswami [24] also evaluated

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(1) Formula (2.16) follows directly from (2.4) upon setting $t = 1/2$ and $\lambda = s$.

a number of special sums including, for example,

(2.20) $$\sum_{k=1}^{\infty} \frac{\zeta(2k+1)}{2k+1} 2^{-2k} = \log 2 - \gamma$$

(2.21) $$\sum_{k=1}^{\infty} \frac{\zeta(2k+1)}{2^{2k+1}} = \log 2 - \frac{1}{2}$$

(2.22) $$\sum_{k=1}^{\infty} \frac{2k-1}{2k+1} \frac{\zeta(2k+1)}{2^{2k+1}} = \gamma - \frac{1}{2}$$

(2.23) $$\sum_{k=1}^{\infty} \frac{1}{(2k+1)(2k+2)} \frac{\zeta(2k)}{2^{2k}} = \frac{1}{4} - \frac{7\zeta(3)}{24\zeta(2)}$$

where γ denotes the Euler-Mascheroni constant defined by

(2.24) $$\gamma = \lim_{n\to\infty} \{1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \log n\} \cong 0.5772156649\ldots.$$

It is easy to verify the fact that any two of the summation formulas (2.20), (2.21) and (2.22) imply the third. Formula (2.20) is contained in a memoir of 1781 by Leonhard Euler (1707-1783) (cf. Glaisher [9], p. 28, Equation (8)); it was rederived by Wilton ([34], p. 92) who showed also that

(2.25) $$\sum_{k=1}^{\infty} \frac{1}{k(2k+1)} \frac{\zeta(2k)}{2^{2k}} = \log \pi - 1$$

(2.26) $$\sum_{k=1}^{\infty} \frac{(2k)!}{(2k+3)!} \frac{\zeta(2k+1)}{2^{2k+1}} = \frac{3}{8} - \frac{1}{6}\log \pi - \frac{1}{4}\gamma + \frac{1}{12}\log 2 + \frac{\zeta'(2)}{\pi^2}$$

(2.27) $$\sum_{k=1}^{\infty} \frac{(2k-1)!}{(2k+3)!} \frac{\zeta(2k)}{2^{2k}} = \frac{\zeta(3)}{2\pi^2} + \frac{1}{12}\log \pi - \frac{11}{72}.$$

Furthermore, since

(2.28) $$\sum_{k=1}^{\infty} \frac{x^{2k-1}}{2k-1} = \frac{1}{2}\log\frac{1+x}{1-x} \qquad |x| < 1$$

so that, for $x = 1/2$,

(2.29) $$\sum_{k=1}^{\infty} \frac{2^{-2k}}{2k+1} = \log 3 - 1$$

the summation formula (2.20) is an immediate consequence of the following result (also contained in Euler's memoir of 1781 already referred to)

$$\sum_{k=1}^{\infty} \frac{\zeta(2k+1)-1}{(2k+1)\,2^{2k}} = 1-\gamma-\log\frac{3}{2} \tag{2.30}$$

which was rederived in 1826 by Legendre ([**19**], p. 434; see also Stieltjes [**28**], p. 302; Glaisher [**9**], p. 28, Equation (9) who recalls both (2.20) and (2.30) erroneously). Legendre ([**19**], p. 434) also showed that

$$\sum_{k=1}^{\infty} \frac{\zeta(2k+1)-1}{2k+1} = 1-\gamma-\frac{1}{2}\log 2\,. \tag{2.31}$$

While presenting alternative (direct) proofs of the summation formulas (2.30) and (2.31), Johnson [**14**] obtained a number of results including, for example, the sum (Johnson [**14**], p. 480, Equation (8); see also Verma and Kaur [**32**], p. 181, Equation (D))

$$\sum_{k=1}^{\infty} \frac{\zeta(2k)-1}{k} = \log 2\,. \tag{2.32}$$

Formulas (2.31) and (2.32), together, imply the well-known result (contained in the aforementioned 1781 memoir by Euler)

$$\sum_{k=2}^{\infty} \frac{\zeta(k)-1}{k} = 1-\gamma \tag{2.33}$$

which has appeared in several subsequent works (see, for example, Glaisher [**9**], p. 28, Equation (4); Johnson [**14**], p. 478, Equation (4); Bromwich [**5**], p. 526, Example 6; Wilton [34], p. 93; Barnes and Kaufman [3], where it is posed as a problem; Verma and Kaur [32], p. 181, Equation (A), where it is rederived in a standard manner).

We conclude this section by recalling the formula (cf. Glaisher [**9**], p. 27, Equation (1); Johnson [**14**], p. 478, Equation (3))

$$\sum_{k=2}^{\infty} \frac{k-1}{k}\{\zeta(k)-1\} = \gamma \tag{2.34}$$

which was given in Euler's memoir of 1769, and also the following results contained in Wilton's work [34]

(2.35) $$\sum_{k=1}^{\infty} \frac{\zeta(2k)}{k(2k+1)} = \log(2\pi) - 1$$ or, equivalently,

(2.36) $$\sum_{k=1}^{\infty} \frac{\zeta(2k)-1}{k(2k+1)} = \log(8\pi) - 3$$ and

(2.37) $$\sum_{k=1}^{\infty} \frac{\zeta(2k)}{2^{2k}k} = \log(\frac{1}{2}\pi)$$ or, equivalently,

(2.38) $$\sum_{k=1}^{\infty} \frac{\zeta(2k)-1}{2^{2k}k} = \log\frac{3\pi}{8}.$$

Obviously, Euler's formula (2.34) follows immediately upon subtracting (2.33) from (1.4). Formulas (2.37) and (2.38), on the other hand, complement the sums (2.20) and (2.30), respectively.

3 - Further consequences of the sums (2.3) and (2.4)

Many of the summation formulas mentioned in the preceding sections would follow readily by suitably specializing the following straightforward consequences of (2.3)

(3.1) $$\sum_{k=0}^{\infty} \binom{\lambda+2k-1}{2k} \{\zeta(\lambda+2k)-1\} t^{2k} = \frac{1}{2}\{\zeta(\lambda,\ 2-t)+\zeta(\lambda,\ 2+t)\} \qquad |t|<2$$

(3.2) $$\sum_{k=0}^{\infty} \binom{\lambda+2k}{2k+1} \{\zeta(\lambda+2k+1)-1\} t^{2k+1} = \frac{1}{2}\{\zeta(\lambda,\ 2-t)-\zeta(\lambda,\ 2+t)\} \qquad |t|<2$$

which are derivable also from (2.18) and (2.19), respectively.

Now we replace the summation index k in (2.3) by $k+2$, set $\lambda = s-1$, and divide both sides of the resulting equation by t^2. We thus find from (2.3) that

(3.3) $$\sum_{k=0}^{\infty} \binom{s+k}{k+2} \{\zeta(s+k+1)-1\} t^k$$
$$= t^{-2}\{\zeta(s-1,\ 2-t) - \zeta(s-1) + 1\} - (s-1)t^{-1}\{\zeta(s)-1\} \qquad 0<|t|<2$$

and (2.4) similarly yields

$$\sum_{k=0}^{\infty}\binom{s+k}{k+2}\zeta(s+k+1)\,t^{k} \tag{3.4}$$

$$=t^{-2}\{\zeta(s-1,\ 1-t)-\zeta(s-1)\}-(s-1)\,t^{-1}\,\zeta(s) \qquad 0<|t|<1.$$

Differentiating both sides of (3.3) and (3.4) with respect to t, and using the formulas (1.11) and (2.9), we obtain

$$\sum_{k=1}^{\infty}\frac{k(s)_{k+1}}{(k+2)!}\{\zeta(s+k+1)-1\}\,t^{k-1} \tag{3.5}$$

$$=t^{-2}\{\zeta(s,\ 2-t)+\zeta(s)-1\}-\frac{2t^{-3}}{s-1}\{\zeta(s-1,\ 2-t)-\zeta(s-1)+1\} \qquad 0<|t|<2$$

and

$$\sum_{k=1}^{\infty}\frac{k(s)_{k+1}}{(k+2)!}\zeta(s+k+1)\,t^{k-1} \tag{3.6}$$

$$=t^{-2}\{\zeta(s,\ 1-t)+\zeta(s)\}-\frac{2t^{-3}}{s-1}\{\zeta(s-1,\ 1-t)-\zeta(s-1)\} \qquad 0<|t|<1$$

respectively.

For $t=-1$, (3.5) readily yields

$$\zeta(s)=1+\frac{1}{2^{s+1}}\frac{s+3}{s-1}+\frac{1}{2}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{k(s)_{k+1}}{(k+2)!}\{\zeta(s+k+1)-1\} \tag{3.7}$$

while (3.6) *formally* reduces, when $t\to-1$, to the sum

$$\zeta(s)=\frac{1}{2}+\frac{1}{s-1}+\frac{1}{2}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{k(s)_{k+1}}{(k+2)!}\zeta(s+k+1) \qquad \mathrm{Re}(s)<1. \tag{3.8}$$

Formula (3.7) follows also from (2.12). As a matter of fact, formulas (3.7) and (3.8) happen to be the main results in a recent paper by Singh and Verma [27] who prove each of these results in a markedly different manner.

By assigning suitable special values to the variable t in (3.5) and (3.6), we can deduce a large number of sums of series involving the zeta function. For example, for $t = 1$, (3.5) immediately yields the result

$$\zeta(s) = \frac{1}{2} + \frac{1}{s-1} + \frac{1}{2} \sum_{k=2}^{\infty} \frac{(k-1)(s)_k}{(k+1)!} \{\zeta(s+k) - 1\} \tag{3.9}$$

which is derivable also from Landau's formula (2.11).

4 - Sums of series involving $f(\zeta(k))/k$

In the theory of Γ-functions, it is fairly well known that (see, e.g., Erdélyi et al. [8], p. 45, Equation 1.17(2); Jordan [15], p. 62, Equation (2))

$$\log \Gamma(1+t) = -\gamma t + \sum_{k=2}^{\infty} (-1)^k \zeta(k) \frac{t^k}{k} \qquad |t| < 1 \tag{4.1}$$

or, equivalently, that (cf. Abramowitz and Stegun [1], p. 256, Equation (6.1.33))

$$\log \Gamma(2+t) = (1-\gamma)\, t + \sum_{k=2}^{\infty} (-1)^k \{\zeta(k) - 1\} \frac{t^k}{k} \qquad |t| < 2. \tag{4.2}$$

For $t \to 1$, (4.1) reduces immediately to the classical result (see Jordan [15], p. 62; Erdélyi et al. [8], p. 45, Equation 1.17(3))

$$\sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k)}{k} = \gamma \tag{4.3}$$

and (4.2) with $t = 1$ yields (cf. Verma [31]; see also Verma and Kaur [32], p. 182, Equation (1))

$$\sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k) - 1}{k} = \gamma - 1 + \log 2 \tag{4.4}$$

which is, of course, equivalent to (4.3).

The special case of (4.2) when $t = -1$ gives us the well-known result (2.33). Formula (2.33) in conjunction with (4.4) would immediately yield the summation formulas (2.31) and (2.32).

Setting $t = \pm 1/2$ in (4.1), we have

$$\sum_{k=2}^{\infty} (-1)^k \zeta(k) \frac{2^{-k}}{k} = \frac{1}{2}\gamma + \frac{1}{2}\log \pi - \log 2 \tag{4.5}$$

$$\sum_{k=2}^{\infty} \zeta(k) \frac{2^{-k}}{k} = \frac{1}{2}\log \pi - \frac{1}{2}\gamma \tag{4.6}$$

or, equivalently,

$$\sum_{k=2}^{\infty} (-1)^k \{\zeta(k) - 1\} \frac{2^{-k}}{k} = \frac{1}{2}(\gamma - 1) + \frac{1}{2}\log \pi + \log \frac{3}{4} \tag{4.7}$$

$$\sum_{k=2}^{\infty} \{\zeta(k) - 1\} \frac{2^{-k}}{k} = \frac{1}{2}(1 - \gamma) + \frac{1}{2}\log \pi - \log 2 . \tag{4.8}$$

Formulas (4.5) and (4.6), together, yield the results (2.20) and (2.37), and the summation formulas (2.30) and (2.38) are similar consequences of (4.7) and (4.8).

Finally, we set $t = \pm 3/2$ in (4.2), and we obtain the sums

$$\sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k) - 1}{k} \left(\frac{3}{2}\right)^k = \log \frac{15}{8} + \frac{1}{2}\log \pi - \frac{3}{2}(1 - \gamma) \tag{4.9}$$

$$\sum_{k=2}^{\infty} \frac{\zeta(k) - 1}{k} \left(\frac{3}{2}\right)^k = \frac{1}{2}\log \pi + \frac{3}{2}(1 - \gamma) . \tag{4.10}$$

With a view to simplifying the derivation of such summation formulas as (2.20), (2.30), (2.31), (2.32), (2.37) and (2.38), we record here the following consequences of (4.1) and (4.2):

$$\sum_{k=1}^{\infty} \zeta(2k) \frac{t^{2k}}{k} = \log \left(\frac{\pi t}{\sin \pi t}\right) \qquad |t| < 1 \tag{4.11}$$

$$\sum_{k=1}^{\infty} \zeta(2k+1) \frac{t^{2k+1}}{2k+1} = \frac{1}{2}\log \left(\frac{\Gamma(1-t)}{\Gamma(1+t)}\right) - \gamma t \qquad |t| < 1 \tag{4.12}$$

$$\sum_{k=1}^{\infty} \{\zeta(2k) - 1\} \frac{t^{2k}}{k} = \log \left(\Gamma(2-t)\Gamma(2+t)\right) \qquad |t| < 2 \tag{4.13}$$

and

$$\sum_{k=1}^{\infty} \{\zeta(2k+1) - 1\} \frac{t^{2k+1}}{2k+1} = \frac{1}{2}\log \left(\frac{\Gamma(2-t)}{\Gamma(2+t)}\right) - (\gamma - 1)t \qquad |t| < 2. \tag{4.14}$$

5 - Sums of series involving $f(\zeta(k))/(k+1)$

Differentiating both sides of (4.1) with respect to t, we obtain (see, e.g., Erdélyi et al. [8], p. 45, Equation 1.17(5); Jordan [15], p. 327, Equation (2))

$$(5.1) \qquad \psi(1+t) = -\gamma + \sum_{k=2}^{\infty} (-1)^k \zeta(k)\, t^{k-1} \qquad |t|<1$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$. Now multiply (5.1) by t and integrate both sides between $t=0$ and $t=z$, and we find that

$$(5.2) \qquad \sum_{k=2}^{\infty} (-1)^k \zeta(k) \frac{z^{k+1}}{k+1} = z \log\Gamma(1+z) + \frac{1}{2}\gamma z^2 - \int_0^z \log\Gamma(1+t)\, dt \qquad |z|<1.$$

In precisely the same manner, we find from (4.2) that

$$(5.3) \qquad \sum_{k=2}^{\infty} (-1)^k \{\zeta(k)-1\} \frac{z^{k+1}}{k+1}$$

$$= z \log\Gamma(2+z) + \frac{1}{2}(\gamma-1)\, z^2 - \int_0^z \log\Gamma(2+t)\, dt \qquad |z|<2.$$

Since ([**10**], p. 661, Entry 6.441(1))

$$(5.4) \qquad \int_p^{p+1} \log\Gamma(q+t)\, dt = \frac{1}{2}\log(2\pi) + (p+q)\{\log(p+q) - 1\}$$

it is readily seen from (5.2) with $z \to 1$ that

$$(5.5) \qquad \sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k)}{k+1} = 1 + \frac{1}{2}\gamma - \frac{1}{2}\log(2\pi)$$

which was proved by Suryanarayana [**29**], and again by Singh and Verma ([**27**], p. 3, Section 4). The method of derivation of (5.5) by these earlier workers is fairly standard in the theory of the Riemann zeta function. By the same method, Suryanarayana ([**29**], p. 143, Equation (14)) claimed to have summed the *obviously divergent* alternating series $\sum_{k=2}^{\infty} (-1)^k \zeta(k)$ whose *corrected* (convergent) version is precisely the same as the well-known result (1.7).

Making use of the elementary integral (5.4), the special case of (5.3) when $z = \pm 1$ can easily be rewritten in the forms

(5.6) $$\sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k)-1}{k+1} = \frac{3}{2} + \frac{1}{2}\gamma - \frac{1}{2}\log(8\pi)$$

(5.7) $$\sum_{k=2}^{\infty} \frac{\zeta(k)-1}{k+1} = \frac{3}{2} - \frac{1}{2}\gamma - \frac{1}{2}\log(2\pi)$$

which, together, yield the following sums

(5.8) $$\sum_{k=1}^{\infty} \frac{\zeta(2k)-1}{2k+1} = \frac{3}{2} - \frac{1}{2}\log(4\pi)$$

(5.9) $$\sum_{k=2}^{\infty} \frac{\zeta(2k-1)-1}{k} = \log 2 - \gamma .$$

Formula (5.6) follows at once upon subtracting the elementary result

(5.10) $$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = \log 2$$

from (5.5). Adding (5.10) to (5.5), we similarly obtain

(5.11) $$\sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k)+1}{k+1} = \frac{1}{2} + \frac{1}{2}\gamma - \frac{1}{2}\log\frac{\pi}{2} .$$

In view of the series (5.10), the summation formulas (5.6) and (5.11) are substantially the same as the known result (5.5). Formulas (5.6) and (5.7) and indeed also the well-known result (2.33), happen to be the main results in a recent paper by Verma and Kaur ([32], p. 181, Equations (A), (B), (C)) who also state an erroneous version of the sum (5.9) above ([32], p. 181, Equation (F)). Furthermore, the summation formulas (5.6) and (5.7) appeared more recently as a problem (see [6]).

Summation formulas like (5.8) would follow more rapidly if we multiply both sides of (5.1) by t and integrate the resulting equation from $t = -z$ to $t = z$. We thus find from (5.1) that

(5.12) $$\sum_{k=1}^{\infty} \zeta(2k) \frac{z^{2k}}{2k+1} = \frac{1}{2}\log\left(\frac{\pi z}{\sin \pi z}\right) - \frac{1}{2z} \int_{-z}^{z} \log \Gamma(1+t)\, dt \qquad 0 < |z| < 1$$

or, equivalently, that (cf. Equation (5.3))

$$\sum_{k=1}^{\infty} \{\zeta(2k)-1\} \frac{z^{2k}}{2k+1} = \frac{1}{2} \log(\Gamma(2+z)\Gamma(2-z)) - \frac{1}{2z} \int_{-z}^{z} \log \Gamma(2+t)\, dt \qquad 0<|z|<2 \tag{5.13}$$

which would follow directly from (4.2) in precisely the same manner as (5.12) follows from (4.1).

As an example of the use of the summation formulas (5.12) and (5.13), we set $z=1/2$ in (5.12) and evaluate the resulting integral by means of (5.4). We thus obtain the sum

$$\sum_{k=1}^{\infty} \zeta(2k) \frac{2^{-2k}}{2k+1} = \frac{1}{2} - \frac{1}{2} \log 2 \tag{5.14}$$

while (5.13) with $z=1/2$ yields the sum

$$\sum_{k=1}^{\infty} \{\zeta(2k)-1\} \frac{2^{-2k}}{2k+1} = \frac{3}{2} - \frac{1}{2} \log 2 - \log 3 . \tag{5.15}$$

In view of the series (2.29), this last summation formula is an immediate consequence of (5.14).

Finally, we subtract the series (5.14) from the series (5.8), and we find that

$$\sum_{k=1}^{\infty} \frac{(1-2^{-2k})\,\zeta(2k)-1}{2k+1} = 1 - \frac{1}{2} \log(2\pi) \tag{5.16}$$

or, equivalently, that (see Robbins [25])

$$\frac{1}{2} \log(2\pi) = 1 - \sum_{m=1}^{\infty} \left\{ \frac{1}{3(2m+1)^2} + \frac{1}{5(2m+1)^4} + \ldots \right\} . \tag{5.17}$$

6 - Miscellaneous results and generalizations

Since

$$\frac{\lambda k+\mu}{k(k+1)} = \frac{\mu}{k} + \frac{\lambda-\mu}{k+1} \tag{6.1}$$

the various summation formulas established in the preceding sections can be applied to deduce sums of series involving, for example, $\frac{f(\zeta(k))}{k(k+1)}$. In particular, the summation formulas (2.33) and (5.7) lead us in this way to the sum

$$(6.2)\qquad \sum_{k=2}^{\infty} \frac{\lambda k+\mu}{k(k+1)}\{\zeta(k)-1\} = \mu - \frac{1}{2}(\lambda+\mu)\gamma + \frac{1}{2}(\lambda-\mu)\{3-\log(2\pi)\}$$

while (4.4) and (5.6) similarly yield the sum

$$(6.3)\qquad \sum_{k=2}^{\infty} (-1)^k \frac{\lambda k+\mu}{k(k+1)}\{\zeta(k)-1\}$$

$$= \mu(\log 2-1) + \frac{1}{2}(\lambda+\mu)\gamma + \frac{1}{2}(\lambda-\mu)\{3-\log(8\pi)\}.$$

By assigning suitable special values to the arbitrary constants λ and μ, we can obtain a number of interesting summation formulas as immediate consequences of (6.2) and (6.3). For instance, the special case of (6.2) when $\lambda=-\mu=1$ yields the known sum (see Chrystal [7], p. 372, Equation (18))

$$(6.4)\qquad \sum_{k=2}^{\infty} \frac{k-1}{k(k+1)}\{\zeta(k)-1\} = 2-\log(2\pi).$$

An alternative method of obtaining sums of series involving $\frac{\zeta(k)}{k(k+1)}$ or $\frac{f(\zeta(k))}{k(k+1)}$ is provided by the well-known formulas (4.1) and (4.2). Indeed, if we merely integrate both sides of (4.1) and (4.2) from $t=0$ to $t=z$, we get

$$(6.5)\qquad \sum_{k=2}^{\infty} (-1)^k \zeta(k) \frac{z^k}{k(k+1)} = \frac{1}{2}\gamma z + \frac{1}{z}\int_0^z \log\Gamma(1+t)\,dt \qquad 0<|z|<1$$

$$(6.6)\qquad \sum_{k=2}^{\infty} (-1)^k \{\zeta(k)-1\} \frac{z^k}{k(k+1)} = \frac{1}{2}(\gamma-1)z + \frac{1}{z}\int_0^z \log\Gamma(2+t)\,dt \qquad 0<|z|<2.$$

On the other hand, by integrating (4.1) and (4.2) from $t=-z$ to $t=z$, we have (cf. [12], p. 356, Equation (54.5.5))

$$(6.7)\qquad \sum_{k=1}^{\infty} \zeta(2k)\frac{z^{2k}}{k(2k+1)} = \frac{1}{z}\int_{-z}^{z} \log\Gamma(1+t)\,dt \qquad 0<|z|<1$$

$$(6.8)\qquad \sum_{k=1}^{\infty} \{\zeta(2k)-1\}\frac{z^{2k}}{k(2k+1)} = \frac{1}{z}\int_{-z}^{z} \log\Gamma(2+t)\,dt \qquad 0<|z|<2$$

which obviously generalize Wilton's formulas (2.35) and (2.36), respectively.

Multiplying (5.3) by λ/z, and (6.6) by μ, and adding the resulting equations, we obtain the following unification (and generalization) of the summation formulas (6.2) and (6.3), and indeed also of (6.6)

$$(6.9)\qquad \sum_{k=2}^{\infty} (-1)^k \frac{\lambda k+\mu}{k(k+1)}\{\zeta(k)-1\}z^k$$

$$= \lambda \log\Gamma(2+z) + \frac{1}{2}(\lambda+\mu)(\gamma-1)z - \frac{\lambda-\mu}{z}\int_{0}^{z} \log\Gamma(2+t)\,dt \qquad 0<|z|<2$$

which, in view of (5.4), would yield (6.2) and (6.3) in its special cases when $z=-1$ and $z=1$, respectively.

In the special case of (6.7) when $z=1/2$, if we evaluate the resulting integral by means of (5.4), we shall obtain Wilton's result (2.25), while (6.8) with $z=1/2$ similarly yields the sum

$$(6.10)\qquad \sum_{k=1}^{\infty} \{\zeta(2k)-1\}\frac{2^{-2k}}{k(2k+1)} = \log\left(\frac{27\pi}{4}\right) - 3\,.$$

Wilton's result (2.25), which was posed as a problem over four decades later (see [11]), follows immediately upon setting $a=1$ in Burnside's formula (cf., e.g., Wilton [34], p. 91, Equation (3); see also Erdélyi et al. [8], p. 48, Equation 1.18(11); Magnus et al. [20], p. 12)

$$(6.11)\qquad \sum_{k=1}^{\infty} \zeta(2k,\ a)\frac{2^{-2k}}{k(2k+1)}$$

$$= \log(2\pi) + (2a-1)\left\{\log\left(a-\frac{1}{2}\right)-1\right\} - 2\log\Gamma(a) \qquad \mathrm{Re}(a) > -\frac{1}{2}$$

involving the generalized zeta function defined by (1.9). As a matter of fact, Wilton [34] rederived Burnside's formula (6.11) as a consequence of the following straightforward generalization of the expansion (2.3)

$$\sum_{k=0}^{\infty} \binom{\lambda+k-1}{k} \{\zeta(\lambda+k,\ a) - a^{-\lambda-k}\}\, t^k = \zeta(\lambda,\ 1+a-t) \qquad |t| < |1+a| \tag{6.12}$$

which is, of course, equivalent to the generalization

$$\sum_{k=0}^{\infty} \binom{\lambda+k-1}{k} \zeta(\lambda+k,\ a)\, t^k = \zeta(\lambda,\ a-t) \qquad |t| < |a| \tag{6.13}$$

of the expansion (2.4).

In terms of the generalized zeta function $\zeta(s,\ a)$, it is also known that (cf. Whittaker and Watson [33], p. 276; see also Gradshteyn and Ryzhik [**10**], p. 1074, Entry 9.532)

$$\sum_{k=2}^{\infty} (-1)^k \zeta(k,\ a) \frac{t^k}{k} = \log \Gamma(a+t) - \log \Gamma(a) - t\psi(a) \qquad |t| < |a| \tag{6.14}$$

or, equivalently, that

$$\sum_{k=2}^{\infty} (-1)^k \{\zeta(k,\ a) - a^{-k}\} \frac{t^k}{k} \tag{6.15}$$

$$= \log \Gamma(1+a+t) - \log \Gamma(1+a) - t\{\psi(a) + a^{-1}\} \qquad |t| < |1+a|.$$

Since $\psi(1) = -\gamma$, (6.14) and (6.15) reduce immediately to (4.1) and (4.2), respectively, upon setting $a = 1$. It should also be remarked in passing that, since (cf. Equation (1.12))

$$\zeta(s,\ a) - a^{-s} \equiv \zeta(s,\ a+1) \qquad \psi(z) + z^{-1} \equiv \psi(z+1)$$

every result like (6.13) and (6.14) can be restated rather trivially in the equivalent forms (6.12) and (6.15), respectively, by merely replacing a by $a+1$.

By employing the rather elementary techniques illustrated fairly fully in this section, and in the preceding sections, we can easily derive appropriate generalizations of the various summation formulas considered in this paper as useful consequences of (6.12), (6.13), (6.14) and (6.15). For the sake of completeness, we choose to record some of these generalizations as follows (see also [**12**], p. 358, Equations (54.11.2), (54.11.3) and (54.11.4))

(6.16)
$$\sum_{k=2}^{\infty}(-1)^k\zeta(k,\ a)\frac{z^k}{k(k+1)}$$
$$=\frac{1}{z}\int_0^z \log\Gamma(a+t)\,\mathrm{d}t-\log\Gamma(a)-\frac{1}{2}z\psi(a)\qquad 0<|z|<|a|$$

(6.17)
$$\sum_{k=1}^{\infty}\zeta(2k,\ a)\frac{z^{2k}}{k(2k+1)}=\frac{1}{z}\int_{-z}^{z}\log\Gamma(a+t)\,\mathrm{d}t-2\log\Gamma(a)\qquad 0<|z|<|a|$$

(6.18)
$$\sum_{k=1}^{\infty}\zeta(2k,\ a)\frac{t^{2k}}{k}=\log\Gamma(a+t)+\log\Gamma(a-t)-2\log\Gamma(a)\qquad |t|<|a|$$

(6.19)
$$\sum_{k=1}^{\infty}\zeta(2k+1,\ a)\frac{t^{2k+1}}{2k+1}=\frac{1}{2}\{\log\Gamma(a-t)-\log\Gamma(a+t)\}+t\psi(a)\qquad |t|<|a|.$$

It is easily seen from (6.14) and (6.16) that

(6.20)
$$\sum_{k=2}^{\infty}(-1)^k\frac{\lambda k+\mu}{k(k+1)}\zeta(k,\ a)z^k=\lambda\log\Gamma(a+z)-\mu\log\Gamma(a)$$
$$-\frac{1}{2}(\lambda+\mu)z\psi(a)-\frac{\lambda-\mu}{z}\int_0^z\log\Gamma(a+t)\,\mathrm{d}t\qquad 0<|z|<|a|.$$

Setting $z=-1$ in (6.20), and evaluating the resulting integral by means of (5.4), we obtain the summation formula

(6.21)
$$\sum_{k=2}^{\infty}\frac{\lambda k+\mu}{k(k+1)}\zeta(k,\ a)=\lambda\log\Gamma(a-1)-\mu\log\Gamma(a)+\frac{1}{2}(\lambda+\mu)\psi(a)$$
$$-\frac{1}{2}(\lambda-\mu)\log(2\pi)-(\lambda-\mu)(a-1)\{\log(a-1)-1\}\qquad |\arg(a-1)|<\pi.$$

For $\lambda=-\mu=1$, this last result (6.21) reduces immediately to Binet's formula (cf. Whittaker and Watson [33], p. 261, Example 18; see also Erdélyi et al. [8],

p. 48, Equation 1.18(10), and Magnus et al. [20], p. 12)

$$(6.22)\qquad \sum_{k=2}^{\infty} \frac{k-1}{k(k+1)} \zeta(k, a)$$

$$= 2 \log \Gamma(a) - (2a-1) \log (a-1) - \log (2\pi) + 2(a-1) \qquad |\arg(a-1)| < \pi$$

which, for $a = 2$, yields (6.4).

Next we apply the formulas (6.17) and (6.18) with a view to deriving the sum

$$(6.23)\qquad \sum_{k=1}^{\infty} \frac{\lambda k + \mu}{k(2k+1)} \zeta(2k, a) z^{2k} = \frac{1}{2} \lambda \{\log \Gamma(a+z) + \log \Gamma(a-z)\}$$

$$- 2\mu \log \Gamma(a) - \frac{\lambda - 2\mu}{2z} \int_{-z}^{z} \log \Gamma(a+t)\, dt \qquad 0 < |z| < |a|.$$

Setting $z = 1/2$ in (6.23), and evaluating the resulting integral by means of (5.4), we deduce the following interesting generalization of several results including, for example, Burnside's formula (6.11)

$$(6.24)\qquad \sum_{k=1}^{\infty} \frac{\lambda k + \mu}{k(2k+1)} \zeta(2k, a) 2^{-2k}$$

$$= \lambda \log \Gamma(a + \tfrac{1}{2}) - 2\mu \log \Gamma(a) - \frac{1}{2}(\lambda - 2\mu) \log (2\pi)$$

$$- \{\lambda a - \mu(2a-1)\} \log (a - \tfrac{1}{2}) + (\lambda - 2\mu)(a - \tfrac{1}{2}) \qquad \mathrm{Re}(a) > -\frac{1}{2}$$

which indeed yields (6.11) for $\lambda = \mu - 1 = 0$.

Finally, we integrate (6.19) from $t = 0$ to $t = z$, and we find that

$$(6.25)\qquad \sum_{k=1}^{\infty} \zeta(2k+1, a) \frac{z^{2k+2}}{(k+1)(2k+1)}$$

$$= \int_{0}^{z} \{\log \Gamma(a-t) - \log \Gamma(a+t)\}\, dt + z^2 \psi(a) \qquad |z| < |a|$$

which, for $z=1$, yields

$$\sum_{k=1}^{\infty} \frac{\zeta(2k+1,\ a)}{(k+1)(2k+1)} \tag{6.26}$$

$$= (a-1)\log(a-1) - a\log a + \psi(a) + 1 \qquad |\arg(a-1)| < \pi$$

where we have made use of the elementary integral (5.4).

This last result (6.26) reduces, when $a \to 1$, to the summation formula

$$\sum_{k=1}^{\infty} \frac{\zeta(2k+1)}{(k+1)(2k+1)} = 1 - \gamma \tag{6.27}$$

which is attributed to Glaisher by Ramanujan [23] (p. 73).

Remark. In addition to the numerous references cited in the preceding sections, the summation formulas (1.4), (1.7), (1.8), (2.32), (2.33) and (4.4) appeared *collectively* in the work of Knopp ([17] p. 271, Exercise 124 (a) to (e), (g) and (h)). As a matter of fact, Knopp has also recorded the interesting sum (cf. [17], p. 271, Exercise 124(f))

$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{\zeta(2k)-1}{k} = \log\left(\frac{\sinh \pi}{2\pi}\right) \tag{6.28}$$

which follows readily in the special case of (4.13) when $t=i$, since

$$\Gamma(2-i)\Gamma(2+i) = \frac{2\pi i}{\sin \pi i} = \frac{2\pi}{\sinh \pi} \tag{6.29}$$

where we have used the familiar property

$$\Gamma(1-z)\Gamma(1+z) = \frac{\pi z}{\sin \pi z} \tag{6.30}$$

of the Γ-functions. Furthermore, in view of the elementary result (5.10), the summation formula (6.28) is also an immediate consequence of (4.11) when $t \to i$.

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Summary

This paper aims at presenting a systematic account of several interesting classes of summation formulas involving series of the Riemann zeta function $\zeta(s)$. *Simple proofs are provided for many useful unifications (and generalizations) of various sums which have received considerable attention in recent works. Analogous results associated with the generalized zeta function* $\zeta(s,\ a)$ *are also investigated.*

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