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On a generalization of paracompact spaces (**)

Introduction

In this paper we generalize the notion of paracompact spaces. A generalization of paracompact space was introduced in [2]. But our definition is still more general and better than the concept introduced in [2] and based on the definition of $[a, b]$-compact spaces [1].

1 - Notation and terminology

Let the letters a, b, m and n denote infinite cardinal numbers with $a<b$ and $[a, b]$ stand for the set of all cardinals m such that $a \leqslant m \leqslant b$. $[a, \infty]$ designates all cardinal numbers m such that $m \geqslant a$. Let $|E|$ denote the cardinal number of a set E and let m^+ denote the first cardinal strictly larger than m.

Let X be a space and $\{A_s\}_{s \in S}$ be a family of subsets of X. $\{A_s\}_{s \in S}$ is called *locally-a* or *locally less than a* if for every $x \in X$, there exists a neighbourhood U of x in X such that $|\{s \in S | U \cap A_s \neq \emptyset\}| < a$. $\{A_s\}_{s \in S}$ is called *point-a* or *point less than a* if for every $x \in X$, $|\{s | x \in A_s\}| < a$. $\{A_s\}_{s \in S}$ is called *star-a* if $|\{s \in S | A_s \cap A_{s_0} \neq \emptyset\}| < a$ for every $s_0 \in S$. A topological space X is called *$[a, b]$-paracompact* (*strongly $[a, b]$-paracompact*, *weakly $[a, b]$-paracompact*) if every open covering $\mathcal{U}$ of X with $|\mathcal{U}| < b$ has a locally-a (star-a, point-a) open refinement respectively. X is said to be $[a, \infty]$-paracompact if it is $[a, b]$-paracompact for all $b \geqslant a$.

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In the definition of $[a, b]$-paracompact spaces and its variants we take open covers with $< b$ (rather than $\leqslant b$) sets have open refinements that are locally-a. The use of $< b$ rather than $\leqslant b$ has the advantage of greater generality, because $\leqslant b$ is the same as $< b^+$ but (for example) $< \aleph_0$ is not the same as $\leqslant b$ for any one b. It would also harmonize better with the definition of locally-a, which uses $< a$ rather than $\leqslant a$.

3 - Properties of $[a, b]$-paracompact spaces

The following properties of $[a, b]$-paracompact spaces can be established easily.

(a) *Every (m, n)-paracompact space* [2] *is $[m^+, n]$-paracompact, but the converse is not true. If X is $[\aleph_0, b]$-paracompact, then it is not (a, b)-paracompact for any $a \leqslant \aleph_0$.*

(b) *Every $[a, b]$-compact space is $[a, b]$-paracompact. The converse is not true generally; for the real line with usual topology is $[\aleph_0, \infty]$-paracompact but not $[\aleph_0, \infty]$-compact.*

(c) *Every $[a, b]$-paracompact space is weakly $[a, b]$-paracompact.*

(d) *Every strongly $[a, b]$-paracompact space is $[a, b]$-paracompact.*

(e) *The topological sum $\sum_{s \in S} X_s$ of spaces is $[a, b]$-paracompact (strongly $[a, b]$-paracompact, weakly $[a, b]$-paracompact) if and only if X_s is $[a, b]$-paracompact (strongly $[a, b]$-paracompact, weakly $[a, b]$-paracompact) for all $s \in S$.*

(f) *X is hereditarily $[a, b]$-paracompact if and only if every open subspace of X is $[a, b]$-paracompact.*

(g) *every space X is $[\omega^+, \infty]$-paracompact, where ω denotes the weight of the space X.*

(h) *Let $\{A_K\}_{k \in K}$ be a family of subsets of X such that $|K| < a$.*
If each A_k is $[a, b]$-paracompact, then $\bigcup_{k \in K} A_k$ is $[a, b]$-paracompact.

One interesting question regarding these spaces is: to what extent $[a, b]$-paracompactness is preserved by closed continuous maps? By perfect maps? This will be a delicate matter, because paracompactness is preserved by closed continuous maps, whereas countable paracompactness is not. However one can prove the following

Theorem. *Suppose* f *is a closed continuous mapping of a space* X *onto an* $[a, b]$*-paracompact space* Y *such that* $f^{-1}(y)$ *is initially* b*-compact* [3] *for each* $y \in Y$. *Then* X *is* $[a, b]$*-paracompact.*

Proof. Let $\mathscr{U} = \{O_\alpha\}_{\alpha \in I}$ be any open covering of X such that $|\mathscr{U}| < b$. Let $\mathscr{F}$ denote the family of all finite subsets of I. Now $|I| < b$ and let $V_\Gamma = Y \setminus f(X \setminus \bigcup_{\alpha \in \Gamma} O_\alpha)$, $\Gamma \in \mathscr{F}$. As f is a closed map V_Γ is open. Let y be any point in Y. Now $\mathscr{U}$ is an open cover of $f^{-1}(y)$. By hypothesis $f^{-1}(y)$ is initially b-compact. Therefore there exists a $\Gamma \in \mathscr{F}$ such that $f^{-1}(y) \subset \bigcup_{\alpha \in \Gamma} O_\alpha$, and $y \in V_\Gamma$ for this and also $f^{-1}(V_\Gamma) \subset \bigcup_{\alpha \in \Gamma} O_\alpha$. Then $\mathscr{V} = \{V_\Gamma\}_{\Gamma \in \mathscr{F}}$ is an open covering of Y and $|\mathscr{V}| < b$. As Y is $[a, b]$-paracompact, there exists a locally-a open refinement $\mathscr{W} = \{W_j\}_{j \in J}$ of $\mathscr{V}$ so that for each $j \in J$ there exists a $\Gamma_j \in \mathscr{F}$ such that $W_j \subset V_{\Gamma_j}$. This means that $f^{-1}(W_j) \subset f^{-1}(V_{\Gamma_j}) \subset \bigcup_{\alpha \in \Gamma_j} O_\alpha$. Let $\mathscr{R} = \{f^{-1}(W_j) \cap O_\alpha\}$. It can easily be seen that $\mathscr{R}$ is a locally-a open refinement of $\mathscr{U}$. Hence X is $[a, b]$-paracompact.

Corollary 1. *If* X *is* $[a, b]$*-paracompact and* Y *is compact, then* $X \times Y$ *is* $[a, b]$*-paracompact.*

This corollary gives a partial answer to another interesting question regarding these spaces viz under what conditions the product of an $[a, b]$-paracompact and $[m, n]$-compact space is $[x, y]$-paracompact. An obvious consequence of this corollary is: if X is $[a, \infty]$-paracompact and Y be compact, then $X \times Y$ is $[a, \infty]$-paracompact. But this statement is false if a is not an infinite cardinal. For example let $X = Y =$ Unit interval. Then X is $[3, \infty]$-paracompact and Y is compact, but $X \times Y$ is not $[3, \infty]$-paracompact (being two dimensional)

Corollary 2. *If* f *is a closed continuous mapping of a space* X *onto an* $[a, b]$*-compact space* Y, *such that* $f^{-1}(y)$ *is initially* b*-compact for each point* $y \in Y$, *then* X *is* $[a, b]$*-compact.*

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