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GIOVANNI CIMATTI (*)

On the problem of the electrical heating of a conductor (**)

1 - Introduction

In this paper we study the steady state potential and temperature distribution inside a conductor taking into account the Thomson effect. To make the paper selfcontained we summarize briefly how the basic equations are derived (see [5] and [6]). In the Thomson model a temperature gradient will by itself produce a flow of electrons, even when no electric field is acting on them. Therefore the current density is given by

$$J = \sigma(\theta)(-\nabla\psi + \beta(\theta)\nabla\theta) \tag{1.1}$$

where $\sigma(\theta)$ is the electric conductivity, $\beta(\theta)$ a given function of the absolute temperature θ and ψ is the electric potential. Define

$$B(\theta) = \int_{\bar\theta}^{\theta} \beta(t)\,\mathrm{d}t \qquad \varphi = B(\theta) - \psi \tag{1.2}$$

where $\bar\theta$ is an arbitrary reference value. In this way (1.1) becomes

$$J = \sigma(\theta)\nabla\varphi\,. \tag{1.3}$$

The vector flow of energy per unit area (including the potential and kinetic energy of the particles which constitute the current) is given by

$$q = \psi J + \delta(\theta) J - \varkappa(\theta)\nabla\theta \tag{1.4}$$

(*) Indirizzo: Dipartimento di Matematica, Università di Pisa, via Buonarroti 2, I-56100 Pisa.

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where $\varkappa(\theta)$ is the thermal conductivity and $\delta(\theta)$ a given function representing the mean kinetic energy of the particles.

The conservation of electric charge and energy requires

$$(1.5) \qquad \nabla \cdot \boldsymbol{J} = 0 \qquad\qquad (1.6) \qquad \nabla \cdot q = 0 .$$

By (1.3) and (1.5) it follows

$$(1.7) \qquad \nabla \cdot (\sigma(\theta)\, \nabla\varphi) = 0 .$$

From (1.4) and (1.6) using (1.5) we have

$$(1.8) \qquad -\nabla \cdot (\varkappa(\theta)\, \nabla\theta) + \tau(\theta) \boldsymbol{J} \cdot \nabla\theta = \frac{J^2}{\sigma(\theta)}$$

where $\tau(\theta)$ is the Thomson's coefficient given in the present notations by

$$(1.9) \qquad \tau(\theta) = \delta'(\theta) + \beta(\theta) .$$

The term on the right hand side of (1.8) represents the heat source due to the Joule effect. The expression $\tau(\theta)\boldsymbol{J} \cdot \nabla\theta$ corresponds to the Thomson effect. We remark taht $\tau(\theta)$ has no definite sign.

The aim of this paper is to prove, using a fixed point argument, the existence of at least one classical solutions of the system (1.7), (1.8) with Dirichlet boundary conditions i.e.

$$(1.10) \qquad \theta = \theta_0 \qquad \varphi = \varphi_0 \qquad \text{on } \partial\Omega$$

where $\partial\Omega$ is the boundary of an open and bounded subset representing the conducting body.

We start with a few preliminary remarks. Since θ is the absolute temperature the minimal hypotheses to make the problem physically reasonable are

$$(1.11) \qquad \sigma(0) = 0 \qquad \sigma(\theta) > 0 \qquad \text{for all } \theta > 0$$

$$(1.12) \qquad \varkappa(0) = 0 \qquad \varkappa(\theta) \geqslant \bar{\varkappa} \qquad \text{for all } \theta \geqslant \theta_m$$

where $\bar{\varkappa}$ is a positive constant and $\theta_m = \min \theta_0$ on $\partial\Omega$. Let $(\theta,\ \varphi)$ be a solution of problem (1.7), (1.8), (1.10) so regular that the classical maximum principle can be applied to (1.8). We have $\theta \geqslant \theta_m$ in $\bar{\Omega}$.

It is useful to define a new scale of temperatures $u = \Gamma(\theta)$ as follows

$$\Gamma(\theta) = \int_0^\theta \varkappa(t)\,dt\,. \tag{1.13}$$

By (1.12) $\Gamma(\theta)$ establishes a one to one correspondence of $[\theta_m, \infty)$ into $[u_m, \infty)$ where of course, $u_m = \Gamma(\theta_m)$. In this way problem (1.7), (1.8), (1.10) can be written in the following simpler form

$$\nabla \cdot (a(u)\nabla\varphi) = 0 \qquad \text{in } \Omega \qquad \varphi = \varphi_0 \text{ on } \partial\Omega \tag{1.14}$$

$$-\Delta u + b(u)\,a(u)\,\nabla\varphi\cdot\nabla u = a(u)|\nabla\varphi|^2 \qquad \text{in } \Omega \qquad u = u_0 \text{ on } \partial\Omega\,. \tag{1.15}$$

2 - The existence theorem

The theorem of this section applies only in two dimensions. Thus we suppose Ω to be an open and bounded subset of R^2 representing the orthogonal cross section of a cylindrical body infinitely long in the axial direction. Let $\partial\Omega$ be the regular boundary of Ω (e.g. $\partial\Omega \in C^2$). Suppose u_0 and φ_0 to be traces on $\partial\Omega$ of $C^2(\bar{\Omega})$-functions and put $u_m = \min u_0$ on $\partial\Omega$. Moreover we assume $a(u) \in C^2(R^1_+)$, $b(u) \in C^2(R^1_+)$ and

$$a_1 \geqslant a(u) \geqslant a_0 > 0 \qquad \text{for all } u \geqslant u_m \tag{2.1}$$

$$|b(u)| \leqslant b_1 \qquad \text{for all } u \geqslant u_m \tag{2.2}$$

$$\int_0^\infty |b(t)|\,dt \leqslant B_1\,. \tag{2.3}$$

Theorem. *There exists at least one classical solution of problem* (1.14), (1.15) *if* (2.1), (2.2) *and* (2.3) *hold true.*

Proof. Let us consider the following auxiliary problem

$$\varphi - \varphi_0 \in H^1_0 \cap L^\infty \qquad \int_\Omega \hat{a}(u)\,\nabla\varphi\cdot\nabla\chi\,dx = 0 \qquad \text{for all } \chi \in H^1_0 \tag{2.4}$$

$$u - u_0 \in H^1_0 \qquad \int_\Omega \nabla u\cdot\nabla v\,dx = \int_\Omega B(u)\,\hat{a}(u)\,\nabla\varphi\cdot\nabla v\,dx - \int_\Omega \varphi\hat{a}(u)\,\nabla\varphi\cdot\nabla v\,dx \qquad \text{for all } v \in H^1_0\,. \tag{2.5}$$

We define $\hat{a}(t) \in C^2(\boldsymbol{R}^1)$ as follows

$$(2.6)\qquad \hat{a}(t) = a(t) \quad \text{for all } t \geq u_m \qquad 2a_1 \geq \hat{a}(t) \geq a_0/2 \quad \text{for all } t \in \boldsymbol{R}^1 .$$

Set $\hat{b}(t) \in C^2(\boldsymbol{R}^1)$ such that

$$(2.7)\qquad \hat{b}(t) = b(t) \quad \text{for all } t \geq u_m \qquad |\hat{b}(t)| \leq 2b_1 \quad \text{for all } t \in \boldsymbol{R}^1 .$$

We put now

$$B(t) = \int_{u_m}^{t} \hat{b}(z)\, dz \qquad \text{for all } t \in \boldsymbol{R}^1 .$$

Using the Schauder fixed point theorem we can prove that (2.4), (2.5) has at least one solution. Define the operator $u = T(w)$ from $L^2(\Omega)$ to $L^2(\Omega)$ with the following linear problems

$$(2.8)\qquad \phi - \phi_0 \in H_0^1 \cap L^\infty \qquad \int_\Omega \hat{a}(w)\, \nabla\phi \cdot \nabla\chi\, dx = 0 \qquad \text{for all } \chi \in H_0^1$$

$$(2.9)\qquad u - u_0 \in H_0^1 \qquad \int_\Omega \nabla u \cdot \nabla v\, dx = \int_\Omega (B(w) - \phi)\, \hat{a}(w)\, \nabla\phi \cdot \nabla v\, dx \qquad \text{for all } v \in H_0^1 .$$

Equation (2.8) can be uniquely solved since $\hat{a}(w) \in L^2(\Omega)$. By the maximum principle we have

$$(2.10)\qquad \inf_{\partial\Omega} \phi_0 \leq \phi \leq \sup_{\partial\Omega} \phi_0 .$$

Moreover putting $\chi = \phi - \phi_0$ in (2.8) we obtain

$$(2.11)\qquad \int_\Omega \hat{a}(w) |\nabla\phi|^2\, dx \leq C_1$$

and by (2.6)

$$(2.12)\qquad \int_\Omega |\nabla\phi|^2\, dx \leq C_2 .$$

By (2.10) and (2.12) we have $(B(w) - \phi)\, \hat{a}(w)\, \nabla\phi \in L^2(\Omega)$. Hence (2.8) has one and only one solution in H^1. Thus T is well-defined. Choosing $v = u - u_0$ in (2.9) we get with simple calculations recalling (2.6), (2.7), (2.10) and (2.11),

$$\int_\Omega |\nabla u|^2 \, dx \leq C_3[1 + (\int_\Omega |\nabla u|^2 \, dx)^{\frac{1}{2}}]. \tag{2.13}$$

Hence it follows

$$\int_\Omega |\nabla u|^2 \, dx \leq C_4$$

moreover applying the Poincarè inequality to $u - u_0$ we obtain

$$\int_\Omega |u|^2 \, dx \leq C_5 \tag{2.14}$$

where the constant C_5 does not depend on the choice of w in L^2. We claim that T is continuous. Suppose $w_n \to w^*$ in L^2 and let (φ^*, u^*) be defined by the problems

$$\varphi^* - \varphi_0 \in H_0^1 \cap L^\infty \qquad \int_\Omega \hat{a}(w^*) \nabla\varphi^* \cdot \nabla\chi \, dx = 0 \qquad \text{for all } \chi \in H_0^1 \tag{2.15}$$

$$u^* - u_0 \in H_0^1 \qquad \int_\Omega \nabla u^* \cdot \nabla v \, dx = \int_\Omega [B(w^*) - \varphi^*] \hat{a}(w^*) \nabla\varphi^* \cdot \nabla v \, dx \qquad \text{for all } v \in H_0^1. \tag{2.16}$$

Let (φ_n, u_n) be the sequence of solutions obtained by putting $w = w_n$ in (2.8) and (2.9). Taking $\chi = \varphi_n - \varphi_0$ we have

$$\int_\Omega |\nabla\varphi_n|^2 \, dx \leq C_5. \tag{2.17}$$

Therefore it is possible to extract from $\{\varphi_n\}$ a subsequence weakly convergent in H^1 and strongly in L^2. If $\bar{\varphi}$ is the limit function letting $n \to \infty$ we find

$$\bar{\varphi} - \varphi_0 \in H_0^1 \cap L^\infty \qquad \int_\Omega \hat{a}(w^*) \nabla\bar{\varphi} \cdot \nabla\chi \, dx = 0 \qquad \text{for all } \chi \in H_0^1.$$

Since the solution of (2.15) is unique, we conclude that the entire sequence $\{\varphi_n\}$ converges to φ^*. Now $(B(w_n) - \varphi_n) \hat{a}(w_n) \nabla\varphi_n$ converges weakly to $(B(w^*) - \varphi^*) \hat{a}(w^*) \nabla\varphi^*$ by (2.6), (2.7) and because by the maximum principle we have $|\varphi_n| \leq \sup |\varphi_0|$ on $\partial\Omega$. It follows by (2.9) that u_n converges to u^* weakly in H^1 and strongly in L^2. This implies the continuity of T. Let Σ be a closed and convex subset of L^2 given by

$$\Sigma = \{v \in L^2;\ \int_\Omega |v|^2 \, dx \leq C_5\}$$

where C_5 is the constant appearing in (2.14). We have $T(\Sigma)\subset\Sigma$, moreover $T(\Sigma)$ is compact. By the Schauder fixed point theorem, we conclude that T has at least one fixed point. Therefore problem (2.8), (2.9) has at least one solution $(\varphi,\ u)$.

We claim that $(\varphi,\ u)$ is in fact a classical solution of problem (1.14), (1.15). From (2.4) it follows by a theorem of Meyers [7], that $\nabla\varphi\in L^{2+\varepsilon}$, $\varepsilon>0$. Hence $(B(u)-\varphi)\,\hat{a}(u)\,\nabla\varphi\in L^{2+\varepsilon}$ and we have $\nabla u\in L^{2+\eta}$, $\eta>0$ again by Meyers' theorem applied this time to equation (2.5). Therefore $\partial\hat{a}(u)/\partial x_i\in L^{2+\eta}$. This permit to apply alternatively to (2.4) and (2.5) the usual bootstrap argument and to conclude that the regularity of $(\varphi,\ u)$ depends only on the degree of regularity of $\hat{a}(t)$, $\hat{b}(t)$. By our hypotheses we obtain $(\varphi,\ u)\in C^{2,\alpha}(\bar{\Omega})$. Integrating by parts in (2.4) we have

$$\nabla\cdot(\hat{a}(u)\,\nabla\varphi)=0 \quad \text{in } \Omega \qquad \varphi=\varphi_0 \text{ on } \partial\Omega\,. \tag{2.18}$$

Again by integration by parts in (2.5) and by (2.18) we obtain

$$-\Delta u=\hat{a}(u)|\nabla\varphi|^2-\hat{b}(u)\,\hat{a}(u)\,\nabla\varphi\cdot\nabla u \quad \text{in } \Omega \qquad u=u_0 \text{ on } \partial\Omega\,. \tag{2.19}$$

Applying the classical maximum principle to (2.19) we infer $u\geq u_m$. Therefore $\hat{a}(u)=a(u)$ and $\hat{b}(u)=b(u)$. We conclude that $(\varphi,\ u)$ is a classical solution of (1.14), (1.15).

Remark. The following weak formulation of equation (1.15)

$$u-u_0\in H_0^1 \qquad \int_\Omega\nabla u\cdot\nabla v\,\mathrm{d}x=\int_\Omega a(u)|\nabla\varphi|^2 v\,\mathrm{d}x-\int_\Omega b(u)\,a(u)\,(\nabla\varphi\cdot\nabla u)v\,\mathrm{d}x \quad \text{for all } v\in H_0^1\cap L^\infty \tag{2.20}$$

appears at first sight the most naturale. However the Stampacchia-Chicco maximum principle [1], [8], at least in its immediate form, does not applies to (2.20).

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Abstract

Using a fixed point argument a theorem of existence is given for the nonlinear boundary value problem governing the electrical heating of a solid conductor. The electrical and thermal conductivities are supposed to be both given functions of the temperature and account is taken of the Thomson effect.

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