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Degree of approximation of continuous functions (**)

1 - Definitions and notations

Let $C_{2\pi}$ be the space of all 2π-periodic and continuous funtions defined on $[-\pi, \pi]$ and let $E_n^q(f; x)$ be the (E, q)-transform of $s_n(f; x)$, the nth partial sum of the Fourier series of f at x. We write $\omega(\beta; f)$ for the modulus of continuity of f (see [8]; p. 42) and suppose, throughout the paper

$$2\delta < \min[\frac{1}{2}\pi, 1/q] \qquad (q>0)$$

We also use the following notations:

$$(1.1) \qquad 2\varphi_x(u) = f(x+u) + f(x-u) - 2f(x)$$

$$(1.2) \qquad P_q^n(u) = (1+q)^{-n}(1+q^2+2q\cos u)^{\frac{1}{2}n}$$

$$(1.3) \qquad g(u) - 1 = (1-q^2\sin^2 u)^{-\frac{1}{2}}(q\cos u) \qquad (qu<1)$$

$$(1.4) \qquad N = \pi(1+q)/n$$

$$(1.5) \qquad b(s) = \tan^{-1}\frac{\sin s}{q+\cos s} \qquad \text{(for any real number } s)$$

$$(1.6) \qquad A = 2q/(\pi(1+q))^2 \qquad (q>0)$$

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(1.7) $$E(n,\ u) = (1+q)^{-n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \sin\left(k+\frac{1}{2}\right) u$$

(1.8) $$t_r(\theta) = \theta + \frac{r\pi}{n} + \sin^{-1}\left(q \sin\left(\theta + \frac{r\pi}{n}\right)\right) \qquad (r = 0,\ 1,\ 2)$$

(1.9) $$R_n = \int_{1/n}^{c_n} t^{-1} \|\varphi(t) - \varphi(t_1)\| P_q^n(t)\, d\theta$$

where $c_n = n^{-\frac{1}{2}} \log n$ and we write throughout t_r for $t_r(\theta)$, t for $t_0(\theta)$ and $\|\cdot\|$ for the sup-norm with respect to x on $[0,\ 2\pi]$.

2 - Introduction

In 1910, Lebesgue [4] proved the following

Theorem A. If $f \in C_{2\pi} \cap \mathrm{Lip}\,\alpha$ $(0 < \alpha \leqslant 1)$, then

(2.1) $$\|s_n(f) - f\| = O\{n^{-\alpha} \log n\}.$$

In 1928, Alexits [1] proved the following along with other results

Theorem B. If $f \in C_{2\pi} \cap \mathrm{Lip}\,\pi$ $(0 < \alpha \leqslant 1)$, then

(2.2) $$\|\sigma_n^r(f) - f\| = O\{n^{-\alpha} \log n\}$$

where $0 < \alpha \leqslant r \leqslant 1$ and $\sigma_n^r(f;\ x)$ is $(C,\ r)$-mean of $s_n(f;\ x)$.

The case $\alpha = r = 1$ was proved by Bernstein [2]. Theorem B was extended by several workers. For example, see Holland, Sahney and Tzimbalario [5], Mohapatra and Chandra [6].

Recently, we [3] have proved the following

Theorem C. If $f \in C_{2\pi} \cap \mathrm{Lip}\,\alpha$ $(0 < \alpha \leqslant 1)$, then

(2.3) $$\|E_n^q(f) - f\| = O(n^{-\frac{1}{2}\alpha}) \qquad (q > 0).$$

The above results taking together raise the problem as to whether or not the estimate of Theorem B can be obtained by using $(E,\ q)$ $(q > 0)$ means in place of

(C, r)-means. In this paper we answer this question in affirmative (see Corollary 1). In fact we prove a more general result. Precisely we prove the following

Theorem. *Let $f \in C_{2\pi}$ and let $M(y) > 0$ be such that*

$$\omega(y; f) = O\{M(y)\} \qquad (y > 0) \tag{2.4}$$

(2.5) $y^{-1}M(y)$ *be non-increasing with $y > 0$. Then*

$$\|E_n^q(f) - f\| = O\{M(1/n) \log n\}. \tag{2.6}$$

Salem and Zygmund [7] demonstrated that the factor $\log n$ can not be dropt in Theorem A even if, in addition to the hypothesis $f \in \mathrm{Lip}\,\alpha$, we suppose that f is of bounded variation. In this paper we show that if f belongs to a suitable sub-class of $\mathrm{Lip}\,\alpha$ $(0 < \alpha \leq 1)$, then the factor $\log n$ can be dropt from the estimate by using (E, q) $(q > 0)$ means of $s_n(f; x)$ (see Corollary 2).

3 - Lemmas

We shall use the following lemmas in the proof of the theorem.

Lemma 1. *Let $0 \leq u \leq \pi$. Then*

$$P_q^n(u) \leq \exp(-Anu^2). \tag{3.1}$$

For its proof, see Chandra [3], Lemma 1.

Lemma 2. *For $n > 4(1+q)$ $(q > 0)$, we have*

$$b(N) > \pi/2n \tag{3.2}$$

and for $0 < \theta < \delta$

$$t_r - t_{r-1} = O(1/N) \qquad (r = 1, 2) \tag{3.3}$$

$$2t_1 - t - t_2 = O(n^{-2})(\theta + \pi/n). \tag{3.4}$$

Proof. We first consider (3.2). Since $n>4(1+q)$, therefore $(q+\cos N)^{-1}\sin N<1$ and hence

$$b(N)=\frac{\sin N}{q+\cos N}-\frac{1}{3}\left(\frac{\sin N}{q+\cos N}\right)^3+\dots>\frac{2}{3}\,\frac{\sin N}{q+\cos N}\geqslant\frac{2\sin N}{3(1+q)}\geqslant\frac{\pi}{2n}.$$

The proof of (3.3) may be obtained by the second mean value theorem. Therefore we consider (3.4). We observe that, by the second mean value theorem,

$$2t_1-t-t_2=(t_1-t)-(t_2-t_1)=\frac{\pi}{n}\,\frac{q\cos u}{(1-q^2\sin^2 u)^{\frac{1}{2}}}-\frac{\pi}{n}\,\frac{q\cos u'}{(1-q^2\sin^2 u')^{\frac{1}{2}}}$$

for some $u\in[\theta,\ \theta+\pi/n]$ and $u'\in[\theta+\pi/n,\ \theta+2\pi/n]$. Hence

$$|2t_1-t-t_2|\leqslant\frac{2\pi}{n}\,\frac{|\cos u-\cos u'|}{(1-q^2\sin^2 u)^{\frac{1}{2}}}=O(n^{-2})(\theta+\pi/n).$$

This completes the proof of the lemma.

Lemma 3. *Let* $0<\theta<\delta$. *Then*

(3.5) $$P_q^n(t_1)\,g(\theta+\frac{\pi}{n})-P_q^n(t)\,g(\theta)=O(\frac{1}{n})\,\{(\theta+\frac{\pi}{n})+n\,\sin t_1\}\,P_q^n(\theta).$$

Proof. By the second mean value theorem,

(3.6) $$P_q^n(t_1)\,g(\theta+\frac{\pi}{n})-P_q^n(t)\,g(\theta)=\frac{\pi}{n}\left(\frac{d}{dy}\,\{g(y)\,P_q^n(t(y))\}\right)_{y=\xi}$$

for some $\xi\in[\theta,\ \theta+\frac{\pi}{n}]$, where $t(y)=t_0(y)$. Now, by elementary but little tedious computation, we get

$$\frac{d}{dy}\,\{g(y)\,P_q^n(t(y))\}=\left(\frac{q(q^2-1)\sin y}{(1-q^2\sin^2 y)^{3/2}}+g^2(y)\frac{n\alpha^2\sin t}{4(1-\alpha^2\sin^2\frac{1}{2}t)}\right)P_q^n(t(y))$$

where $\alpha=2\sqrt{q}/(1+q)$. We observe that $1-\alpha^2\sin^2\frac{1}{2}t$ is equal to $\cos^2 y$ for $q=1$ and greater than $1-\alpha^2$ for $q\neq 1$. Hence the right hand side of (3.6) does not exceed $O(1/n)\{(\theta+\pi/n)+n\sin t_1\}\, P_q^n(\theta)$ since $P_q^n(y)$ decreases and $\sin y$ increases with y increases. Hence we get the proof of the lemma.

Lemma 4. *Let* (2.4) *and* (2.5) *hold. Then*

$$\|\mathcal{F}\| = O\{M(1/n)\} + O(R_n) \tag{3.7}$$

where

$$\mathcal{F} = \int_{b(N)}^{b(\varepsilon)} \left\{\frac{(t_1-t)\,\varphi_x(t)\,P_q^n(t)\,g(\theta)}{tt_1} - \frac{(t_2-t_1)\,\varphi_x(t_1)\,P_q^n(t_1)\,g(\theta+\pi/n)}{t_1 t_2}\right\} \sin n\theta\,\mathrm{d}\theta\,.$$

Proof. We have

$$\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2 + \mathcal{F}_3 + \mathcal{F}_4 + \mathcal{F}_5 \qquad \text{where:}$$

$$\mathcal{F}_1 = \int_{b(N)}^{c_n} \frac{(t_2-t_1)}{t_1 t_2}\{\varphi_x(t)-\varphi_x(t_1)\}\, P_q^n(t_1)\, g(\theta+\frac{\pi}{n})\,\sin n\theta\,\mathrm{d}\theta$$

$$\mathcal{F}_2 = \int_{c_n}^{b(\varepsilon)} \frac{(t_2-t_1)}{t_1 t_2}\{\varphi_x(t)-\varphi_x(t_1)\}\, P_q^n(t_1)\, g(\theta+\frac{\pi}{n})\,\sin n\theta\,\mathrm{d}\theta$$

$$\mathcal{F}_3 = \int_{b(N)}^{b(\varepsilon)} \frac{(t_2-t_1)}{tt_1}\{P_q^n(t)\,g(\theta) - P_q^n(t_1)\,g(\theta+\frac{\pi}{n})\}\,\varphi_x(t)\,\sin n\theta\,\mathrm{d}\theta$$

$$\mathcal{F}_4 = \int_{b(n)}^{b(\varepsilon)} \frac{(t_2-t_1)(t_2-t)}{tt_1 t_2}\, P_q^n(t_1)\, g(\theta+\frac{\pi}{n})\,\varphi_x(t)\,\sin n\theta\,\mathrm{d}\theta$$

$$\mathcal{F}_5 = \int_{b(N)}^{b(\varepsilon)} \frac{(2t_1-t-t_2)}{tt_1}\, P_q^n(t)\, g(\theta)\,\varphi_x(t)\,\sin n\theta\,\mathrm{d}\theta\,.$$

Now, since $P_q^n(u)$ decreases with u, it is clear by (3.1) that, $\|\mathcal{F}_1\| = O(R_n)$. And by (2.4) and (3.2), we get

$$\|\mathcal{F}_2\| = O(1)\int_{c_n}^{b(\varepsilon)} t_1^{-1} M(t_1)\,\exp(-Ant_1^2)\,\mathrm{d}\theta$$

$$= O\{M(1/n)\}\int_{c_n}^{\varepsilon} \frac{1}{\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}(-\exp(-An\theta^2))\,\mathrm{d}\theta = O\{M(1/n)\}\,.$$

However, (3.3), (3.5) and (2.4) yield that

$$\|\mathcal{T}_3\| = O(1/n) + O(1) \int_{b(N)}^{b(\varepsilon)} M(t)\, P_q^n(\theta)\, d\theta$$

$$= O(1/n) + O(1) \int_{b(N)}^{b(\varepsilon)} t^{-1} M(t)\, t \exp(-An\theta^2)\, d\theta = O\{M(1/n)\}$$

by (2.5), (3.2) and Lemma 1. Also (3.2) and (3.3) yield that

$$\|\mathcal{T}_4\| = O(n^{-2}) \int_{1/n}^{\varepsilon} t^{-3} \|\varphi(t)\|\, d\theta = O\{M(1/n)\}$$

by (2.4) and (2.5). Finally, by (3.2), (3.4) and Lemma 1, we get

$$\|\mathcal{T}_5\| = O(n^{-2}) \int_{1/n}^{\varepsilon} (\theta + \pi/n)\, t_1^{-1} P_q^n(t)\, t^{-1} \omega(t;\, f)\, d\theta$$

$$= O(n^{-2}) \int_{1/n}^{\varepsilon} \exp(-An\theta^2)\, \theta^{-1} M(\theta)\, d\theta = O\{M(1/n)\}\,.$$

Now collecting the estimates, we get (3.7).

4 - Proof of the Theorem

We have

$$E_n^q(f;\, x) - f(x) = \int_0^{\pi} \frac{\varphi_x(u)}{\pi \sin(u/2)} E(n,\, u)\, du = \int_0^N + \int_N^{\varepsilon} + \int_{\varepsilon}^{\pi} = I_1 + I_2 + I_3\,.$$

Then

$$\|E_n^q(f) - f\| \leqslant \|I_1\| + \|I_2\| + \|I_3\|\,. \tag{4.1}$$

Now, we get, by using the inequality $\pi \sin(u/2) \geqslant u$ $(0 \leqslant u \leqslant \pi)$,

$$\|I_1\| \leqslant \int_0^N u^{-1} \omega(u;\, f) |E(n,\, u)|\, du = O\{M(1/n)\} \tag{4.2}$$

by (2.4) and (2.5). Also, by Lemma 1, (2.4) and (2.5), we get

$$\|I_3\| \leqslant \int_{\varepsilon}^{\pi} u^{-1} \omega(u;\, f) \exp(-Anu^2)\, du = O\{M(1/n)\}\,. \tag{4.3}$$

Finally, since $\sin(n+\frac{1}{2}u) = \sin(\frac{1}{2}u)(\cos nu + \sin nu \cot(\frac{1}{2}u))$ and $\cot(\frac{1}{2}u) = O(u) + 2/u$ $(0 < u < \pi/4)$, we get

$$\pi\|I_2\| \leqq \int_{1/n}^{\delta} \omega(u; f) P_q^n(u)\, du \tag{4.4}$$

$$+ \Big\| \int_N^{\delta} \varphi_x(u) \cot(\tfrac{1}{2}u)\{(1+q)^{-n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \sin nu\}\, du \Big\|$$

$$= O\{M(1/n)\} + \|J\|$$

where

$$J = 2 \int_N^{\delta} u^{-1} \varphi_x(u) P_q^n(u) \sin\{n \tan^{-1}(\frac{\sin u}{q+\cos u})\}\, du.$$

Now, by using the transformation $u = t = t(\theta)$, we get

$$du = g(\theta)\, d\theta \quad \text{and} \quad (q+\cos u)^{-1} \sin u = \tan\theta.$$

Hence

$$J = 2 \int_{b(N)}^{b(\delta)} t^{-1} \varphi_x(t) P_q^n(t) g(\theta) \sin n\theta\, d\theta$$

$$= \Big(\int_{b(N)}^{b(\delta)} + \int_{b(N)+\frac{\pi}{n}}^{b(\delta)+\frac{\pi}{n}} + \int_{b(N)}^{b(N)+\frac{\pi}{n}} - \int_{b(\delta)}^{b(\delta)+\frac{\pi}{n}} \Big) (t^{-1} \varphi_x(t) P_q^n(t) g(\theta) \sin n\theta\, d\theta)$$

$$= \int_{b(N)}^{b(\delta)} \{t^{-1} \varphi_x(t) P_q^n(t) g(\theta) - t_1^{-1} \varphi_x(t_1) P_q^n(t_1) g(\theta + \frac{\pi}{n})\} \sin n\theta\, d\theta$$

$$+ \Big(\int_{b(N)}^{b(N)+\frac{\pi}{n}} - \int_{b(\delta)}^{b(\delta)+\frac{\pi}{n}} \Big) (t^{-1} \varphi_x(t) P_q^n(t) g(\theta) \sin n\theta\, d\theta)$$

$$= J_1 + J_2 - J_3.$$

However, by (2.4) and (2.5), we get

$$\|J\| = \|J_1\| + O\{M(1/n)\}. \tag{4.5}$$

We now consider $\|J_1\|$. We first observe that

$$\text{(4.6)} \qquad \|J_1\| \leqslant \Big\| \int_{b(N)}^{b(\varepsilon)} \{t^{-1}\varphi_x(t) - t_1^{-1}\varphi_x(t_1)\} P_q^n(t) g(\theta) \sin n\theta \, d\theta \Big\|$$

$$+ \Big\| \int_{b(N)}^{b(\varepsilon)} \{P_q^n(t) g(\theta) - P_q^n(t_1) g(\theta + \frac{\pi}{n})\} t_1^{-1} \varphi_x(t_1) \sin n\theta \, d\theta \Big\|$$

$$= \|J_{1,1}\| + \|J_{1,2}\| .$$

Proceeding as in $\mathcal{T}_3$ of Lemma 4, we may obtain that

$$\text{(4.7)} \qquad \|J_{1,2}\| = O\{M(1/n)\}$$

and

$$\text{(4.8)} \qquad \|J_{1,1}\| \leqslant \int_{b(N)}^{b(\varepsilon)} t_1^{-1} \|\varphi(t) - \varphi(t_1)\| P_q^n(t) \, d\theta$$

$$+ \Big\|\Big(\int_{b(N)}^{b(N)+\frac{\pi}{n}} + \int_{b(N)+\frac{\pi}{n}}^{b(\varepsilon)} \Big) (\varphi_x(t) \frac{t_1 - t}{tt_1} P_q^n(t) g(\theta) \sin n\theta \, d\theta) \Big\|$$

$$= L_1 + \|L_2 + L_3\| .$$

However, by (3.2), (3.3), (2.4) and (2.5),

$$\text{(4.9)} \qquad \|L_2\| = O(1) \int_{b(N)}^{b(N)+\frac{\pi}{n}} t^{-1} \omega(t; f) g(\theta) \, d\theta = O\{M(1/n)\} .$$

$$\text{(4.10)} \qquad 2\|L_3\| = \Big\| \int_{b(N)}^{b(\varepsilon)} + \int_{b(N)+\frac{\pi}{n}}^{b(\varepsilon)+\frac{\pi}{n}} - \int_{b(N)}^{b(N)+\frac{\pi}{n}} - \int_{b(\varepsilon)}^{b(\varepsilon)+\frac{\pi}{n}} \Big\|$$

$$\leqslant \Big\| \int_{b(N)}^{b(\varepsilon)} + \int_{b(N)+\frac{\pi}{n}}^{b(\varepsilon)+\frac{\pi}{n}} \Big\| + \|L_2\| + \int_{b(\varepsilon)}^{b(\varepsilon)+\frac{\pi}{n}} \omega(t; f) \frac{t_1 - t}{tt_1} g(\theta) \, d\theta$$

$$= \|\mathcal{T}\| + \|L_2\| + O\{M(1/n)\}$$

where $\mathcal{T}$ is as defined in Lemma 4. Now, collecting (4.5) through (4.10), we get

$$\text{(4.11)} \qquad \|J\| = O\{M(1/n)\} + L_1 + \|\mathcal{T}\|$$

where, by Lemma 4,

$$\|\mathcal{T}\| = O\{M(1/n)\} + O(R_n) \tag{4.12}$$

and by (2.4) and (2.5)

$$L_1 = O(R_n) + 2 \int_{c_n}^{\delta} t_1^{-1}\,\omega(t_1;\, f)\, P_q^n(t)\,\mathrm{d}\theta \tag{4.13}$$

$$= O(R_n) + O(1) \int_{c_n}^{\delta} \theta^{-1} M(\theta)\, P_q^n(\theta)\,\mathrm{d}\theta = O(R_n) + O\{M(1/n)\}$$

arguing as in $\|\mathcal{T}_2\|$ of Lemma 4. Combining (4.11), (4.12) and (4.13), we get

$$\|J\| = O(R_n) + O\{M(1/n)\} \tag{4.14}$$

and combining (4.14) with (4.1) through (4.4), we get

$$\|E_n^q(f) - f\| = O(R_n) + O\{M(1/n)\}\,. \tag{4.15}$$

Finally, we oberve that

$$\|\varphi(t_1) - \varphi(t)\| \leqslant \omega(|t_1 - t|;\, f) = O\{M(1/n)\}$$

by (2.5). Thus

$$R_n = O\{M(1/n)\} \int_{1/n}^{c_n} P_q^n(\theta)\,\theta^{-1}\,\mathrm{d}\theta = O\{M(1/n)\,\log n\}\,. \tag{4.16}$$

Combining (4.16) with (4.15), we get (2.6). This completes the proof of the theorem.

5 - Corollaries

We deduce two corollaries from the theorem.

Corollary 1. *Let* $f \in C_{2\pi} \cap \mathrm{Lip}\,\alpha$ $(0 < \alpha \leqslant 1)$. *Then*

$$\|E_n^q(f) - f\| = O\{n^{-\alpha}\log n\} \qquad (0 < \alpha \leqslant 1)\,. \tag{5.1}$$

Assume $M(y) = y^{\alpha}$ $(0\,\alpha \leqslant 1)$ in (2.4). Then (2.5) holds and the corollary follows from (2.6).

Corollary 2. *Let* $f \in C_{2\pi} \cap \mathrm{Lip}\,\alpha$ $(0<\alpha\leq 1)$ *and let* $R_n = O(n^{-\alpha})$ $(0<\alpha\leq 1)$. *Then*

$$\|E_n^q(f) - f\| = O(n^{-\alpha}) \qquad (0<\alpha\leq 1). \tag{5.2}$$

Since $f \in \mathrm{Lip}\,\alpha$ $(0<\alpha\leq 1)$ implies that

$$\omega(y; f) = O(y^\alpha).$$

therefore on letting $M(y) = y^\alpha$ in (4.15), we get Corollary 2.

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Abstract

Generalising an earlier result due to the present author [3], *he has shown as a particular case that the degree of approximation of functions* $f \in \mathrm{Lip}\,\alpha$ $(0<\alpha\leq 1)$ *by* (E, q)*-means of its Fourier series in sup-norm is* $O\{n^{-\alpha}\log n\}$.

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