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On the breaking waves in a channel or in a fluid filled elastic tube with arbitrary cross-sectional area (**)

1 - Introduction

In recent years a great deal of attention has been devoted to studying the nonlinear wave breaking in channels.

The work of Gurtin [3] was particularly significant since he used a simple argument to find the condition for the breaking of a finite amplitude water wave. Gurtin's result was extended by Jeffrey and Muvgi [5] to the case of a rectangular channel of variable width and depth.

Such a problem has been considered also for fluid filled elastic tubes in the special case of an initially uniform tube (in which both the internal area and the speed of propagation of the wavefront are constant at constant pressure), in Moodie and Haddow [8] by the method of characteristics and in Tait and Moodie [12] within Jeffrey's method $[4]_2$.

This paper concerns with the extension of Gurtin's criteria for the breaking of a nonlinear weak discontinuity wave in channels or in fluid filled elastic tubes of arbitrary cross-sectional area.

In 2 we consider the general system of equations for one-dimensional waves in a channel of cross section of arbitrary form [11].

By means of a formal substitution of variables [2], we show that in the

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mathematical model we are considering, are included also the basic governing equations of fluid filled elastic tubes [6].

Within the well established theory of nonlinear hyperbolic waves [1], we point out the general conditions for which a wave propagating along a channel (or tube) breaks after a finite time has elapsed so that a shock formation is possible [13].

In the last part of the paper, taking into account possible constitutive laws obtained in [2], we consider some examples.

2 - Basic equations

We consider the motion of an inviscid, incompressible fluid of constant density in channels of cross section of arbitrary form with two-dimensional bottom topography: $z = -h(x, y)$, where z is positive upward.

If trasversal velocities and, more important, vertical accelerations are neglected then the problem becomes one of long waves in shallow water.

This means that we consider straigt channels only and that we require the width of the channel slowly varies.

As usual, let the x-axis lie in the equilibrium of the water in the direction of propagation, t to represent the time, k the constant gravitational acceleration, $\eta(x, t)$ the elevation of the water above the equilibrium level and $u(x, t)$ the x-component of the velocity.

From the assumption of a channel of arbitrary form, the cross-sectional area will be a function of x and η

$$S = S(x, \eta) . \tag{2.1}$$

The governing equations are given by the conservation of mass equation and the equation of motion in the x-direction (see [11], p. 455)

$$S_t + (Su)_x = 0 \tag{2.2}$$

$$u_t + k\eta_x + uu_x = 0 \tag{2.3}$$

here and in what follows the subscripts denote partial derivatives with respect to the indicated variables.

Since we assume a cross section of arbitrary form then the local speed of surface wave propagation is (see next section)

$$c(x,\ \eta)=(k\frac{S}{S_\eta})^{1/2}. \tag{2.4}$$

Naturally we recover the classical resalt ([11], p. 292) of a local speed of surface wave propagation proportional to the square root of the depth of the water from (2.4) if we assume the channel with rectangular form, i.e. $S(x,\ \eta)=W(x)[\eta+h(x)]$ where $W(x)$ is the width of the channel and $h(x)$ represent the constant depth of the water at fixed x.

Before proceeding further, we notice that by substituting formally $1/k$ by the constant density and η the pressure difference between the inside and outside of the tube, the system of equations (2.2) and (2.3) specializes to the governing model of thin-walled filled elastic tubes.

3 - Weak discontinuity wave amplitude

We assume that at $x=t=0$ the wave proceeds into the positive x-axis direction and that it moves into fluid at rest.

Moreover we assume across the wavefront that, [3]:

(i) u and η are continuous;
(ii) the first derivatives of u and η possess at most jump discontinuities;
(iii) $u(x,\ t)=0$ and $\eta(x,\ t)=\eta_0=$ constant, ahend of the wave.

System of equations (2.1), (2.2), (2.3) may be written in the following matrix form

$$\boldsymbol{U}_t+A\boldsymbol{U}_x+\boldsymbol{B}=0 \tag{3.1}$$

where

$$\boldsymbol{U}=\begin{bmatrix}\eta\\ u\end{bmatrix}\quad A=A(\boldsymbol{U},\ x)=\begin{bmatrix}u & \dfrac{S}{S_\eta}\\ k & u\end{bmatrix}\quad \boldsymbol{B}=\boldsymbol{B}(\boldsymbol{U};\ x)=\begin{bmatrix}u\ \dfrac{S_x}{S_\eta}\\ 0\end{bmatrix}. \tag{3.2}$$

The eigenvalue of the matrix A and the corresponding left and right eigenvectors are

$$\lambda^{\pm}=u\pm c \tag{3.3}$$

where $c(x, \eta) = (k \frac{S}{S_\eta})^{1/2}$ is the local speed of surface wave propagation

$$\boldsymbol{l} = (c, \ \pm c^2/k) \qquad \boldsymbol{d} = (c, \ \pm k)^T. \tag{3.4}$$

Since $S > 0$, for the hyperbolicity of system (3.1) we have to assume $S_\eta > 0$ $[4]_2$.

Taking into account the well known results obtained in [**1**], in the case of propagation in a constant state $U_0 = (\eta_0, \ 0)$ which of interest here the transport equation ruling the wave amplitude is given by

$$\boldsymbol{l}_0 \{ \frac{\mathrm{d}\boldsymbol{\pi}}{\mathrm{d}\sigma} + \varphi_x \boldsymbol{\pi} (\nabla \lambda)_0 \boldsymbol{\pi} \} + (\nabla B_1)_0 \boldsymbol{\pi} = 0 \tag{3.5}$$

where $\varphi = \int \frac{\mathrm{d}x}{\lambda_0(x)} - t$ is a solution of the characteristic equation $\varphi_t + \lambda_0(x) \varphi_x = 0$ and $B_1 = \boldsymbol{l}\boldsymbol{B}$, the index $_0$ indicates the value of any quantity in the constant state and $\mathrm{d}/\mathrm{d}\sigma = \partial_t + \lambda_0(x) \partial_x$ is the derivative along the ray $\mathrm{d}x/\mathrm{d}\sigma = \lambda_0 = c_0$. A superscript $(-)$ will be used to refer to values immediately behind the wavefront in the disturbed region. Similarly, a superscript $(+)$ will refer to the undisturbed state ahead of the wavefront. So

$$\boldsymbol{\pi} \varphi_x = \boldsymbol{U}_x^- - \boldsymbol{U}_x^+ \qquad \nabla = (\frac{\partial}{\partial \eta}, \ \frac{\partial}{\partial u}).$$

However we have $\varphi_x = 1/\lambda_0$ and taking into account that

$$\boldsymbol{\pi} = \pi \boldsymbol{d}_0 \tag{3.6}$$

the equation (3.5), after some manipulation, reduces to

$$\frac{\mathrm{d}\pi}{\mathrm{d}x} + \{ \frac{3}{4} \frac{S_x}{S} - \frac{1}{4} \frac{(S_\eta)_x}{S_\eta} \}_0 \pi + \frac{1}{2} \{ \frac{3S_\eta - SS_{\eta\eta}}{SS_\eta} \}_0 \pi^2 = 0 \, . \tag{3.7}$$

In this case we have $\pi = \eta_x^-$ so that the amplitude of the weak discontinuity wave is such that (see [3]): $\pi = \pi(x)$.

Equation (3.7) is a Bernoulli type equation for $\pi(x)$.

The solution of this equation

$$\pi(x) = \frac{\pi^*(\frac{S^*}{S_0})^{3/4}(\frac{S^*_\eta}{S_{\eta|\eta=\eta_0}})^{-1/4}}{1+\pi^* I(x)} \tag{3.8}$$

where $\pi^* = \pi(0)$ is the arbitrarily prescribed initial condition, $S^* = S(0, \eta_0)$, $S^*_\eta = S_\eta(0, \eta_0)$ and

$$I(x) = \frac{1}{2} S^{*3/4} S^{*-1/4}_\eta \int_0^x \{\frac{3S^2_\eta - SS_{\eta\eta}}{S^{7/4} S^{3/4}_\eta}\}_0 \, ds . \tag{3.9}$$

A wave of depression corresponds to $\pi^* > 0$, where as a wave of elevation to $\pi^* < 0$.

The wave breaking for some $x = x_b$ where $\pi(x_b) = \infty$ so that the water surface behind the wavefront becomes vertical (or the pressure of fluid is here discontinuous).

Hereafter we will examine the possible situations related to the shock wave formation.

We denote by $x = L$ the shore line (or end part of tube).

(a) Let us assume $I(x) > 0$. A wave of depression ($\pi^* > 0$) in a channel (or tube) of arbitrary form can only break if the cross-sectional area of the fluid selves to zero, where $x = L$, only if $I(L) < \infty$. A wave of elevation ($\pi^* < 0$) in a channel always breaks at $x = x_b$ given by $1 + \pi^* I(x_b) = 0$.

If the cross-sectional area of the fluid selves to zero at the shore line (or end part of tube), a wave of elevation propagating towards the shore will break before reaching the shore line if $|\pi^*| > 1/I(L)$, and at the shore line if $|\pi^*| < 1/I(L)$.

(b) If $I(x) = 0$, then the wave can only break at the shore line (or end part of tube) where $S(L, \eta_0) = 0$.

(c) If $I(x) < 0$ we proceed in a similar way as in case (a).

We consider now the non-breaking condition $I(x) = 0$ and we note that a sufficient but not necessary condition for non-breaking is the following partial differential equation

$$3S^2_\eta - SS_{\eta\eta} = 0 . \tag{3.10}$$

Thus integration of (3.10) gives (we remark that $S_\eta > 0$) $S_\eta = \theta(x)^{-2} S^3$ whose integral is

$$S(x, \eta) = \theta(x) \{-2\eta + r(x)\}^{-1/2} \tag{3.11}$$

where $\theta(x)$ and $r(x)$ are arbitrary functions.

We remark that (3.11) is the class of constitutive laws for which the weak discontinuity wave certainly does'nt break after a finite time.

Moreover in [2] there has been given a mathematical approach, within the group analysis, for characterizing classes of constitutive laws of the type

$$S = \theta(x) \{\alpha\eta + r(x)\}^{1/\alpha} \qquad \alpha \neq 0 . \tag{3.12}$$

We note that (3.11) follows by (3.12) for $\alpha = -2$.

We remark that for the classes of constitutive laws (3.12) we have $I(x) > 0$ if $\alpha > -2$ and $I(x) < 0$ if $\alpha < -2$.

4 - Some examples

The breaking waves criteria shown in the above section can be easily applied when physical or mathematical considerations related with a physics fenomena lead to a particular choice of the cross-sectional area functional form. Since here we do not study a particular case but we wont to stay in the most general framework in what follows we will show how, from the given theory, it is possible to recover and extend well known resalts.

At first we note that thin-walled elastic tubes filled with incompressible fluid represent the model usually considered for study of flows in large blood vessels [10].

Some investigators [7]-[10], [12] assume that the undisturbed cross-sectional area and the elastic properties of the tube are independent on x therefore $S = S(\eta)$.

In other words they require that the tube is uniform and the system (3.1) becomes omogeneous.

We rewrite this equation as follows

$$S = S_1 \exp \int F(\eta - \eta_0)\, d\eta \qquad S_1 > 0 \tag{4.1}$$

where S_1 is a constant.

Of course we must require $F>0$ in order to have the hyperbolicity.
We define

$$H=\left[\frac{2F^2-F_\eta}{F}\right]_{\eta=\eta_0}=\text{const}. \tag{4.2}$$

The wave breaks when $H>0$ if $\pi^*<0$ at

$$x_b=-\frac{2}{H\pi^*} \qquad t_b=\frac{1}{c_0}x_b. \tag{4.3}$$

while if $H<0$ the wave breaks when $\pi^*>0$ again at x_b and t_b given by (4.3).

By (4.2) and (4.3) it is possible to deduce breaking point once the function $F(\eta-\eta_0)$ and the value π^* are given explicitly.

So it is a simple matter to recover known as well as new resalts.

For example linear elastic material [7] is characterized by

$$S=S_0\left[1+k\frac{\eta-\eta_0}{2c_0^2}\right]^2. \tag{4.4}$$

(4.4) may be obtained from (4.1) by means of $S_1=S_0$ and

$$F=\frac{k}{c_0^2}\frac{1}{1+k\dfrac{\eta-\eta_0}{2c_0^2}}. \tag{4.5}$$

Since here $I(x)>0$ we consider $\pi^*<0$. By (4.3) it is simple matter to see that breaking occurs when

$$x_b=-\frac{4}{5}\frac{S_0}{\pi^* S_{\eta|\eta=\eta_0}} \qquad t_b=-\frac{4}{5}\frac{S_0}{c_0\pi^* S_{\eta|\eta=\eta_0}}. \tag{4.6}$$

Moreover, for water waves in channels, we consider the special case of (3.12) in which

$$S(x,\ \eta)=\omega[\alpha\eta+r(x)]^{1/\alpha} \qquad \alpha\neq 0 \tag{4.7}$$

where $r(x)$ is the (max.) depth of the channel and ω is a positive constant.

Since the condition $I(x) > 0$ is always satisfied for $\alpha > -2$ we will consider a wave of elevation, i.e. $\pi^* < 0$.

Furthermore we assume that

$$r(x) = \alpha(h - mx)\,. \tag{4.8}$$

The condition in order to have the wave break $1 + \pi^* I(x_b) = 0$ gives us

$$x_b = \frac{h}{m}\left[1 - \left(\frac{2\pi^*}{2\pi^* - m}\right)^{\frac{4\alpha}{2+\alpha}}\right] \tag{4.9}$$

and the related time

$$t_b = \frac{1}{c_0(x_b)}\,x_b\,. \tag{4.10}$$

From (4.9) and (4.10) by setting $\alpha = 1$ it is a simple matter to recover the Jeffrey's results $[4]_1$, [5].

In the case of symmetric triangular channel with variable cross section, in which the angles are not dependent upon x, the relations (4.9) and (4.10) are still valid with $\alpha = 1/2$ and $\omega = 1/4\,\mathrm{tg}\,\gamma$ (γ is angle between the free surface and the bank of the channel).

If the channel has the parabolic shape (characterized by $z = ay^2 - r(x)$ with $a > 0$), the previous considerations hold with $\alpha = 2/3$ and $\omega = 4/3\,\alpha^{1/\alpha}/a^{1/2}$.

In closing, it is worth noting that in the special case where $2\pi^* = m < 0$, we have the non-breaking of finite amplitude water waves of elevation for any $\alpha > -2$ (see $[4]_1$, [5] for $\alpha = 1$).

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Sunto

L'evoluzione di onde di discontinuità viene studiata in canali o in sottili-fluidi tubi elastici di sezione di forma arbitraria. Sono sviluppati i criteri generali per la formazione di onde d'urto. Nella parte finale del lavoro sono considerati alcuni esempi.

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