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Common fixed points of a family of mappings in Menger and uniform spaces (**)

Introduction

Sehgal and Bharucha-Reid [7] extended the notion of contractions and local contractions to the setting of Menger spaces and obtained some fixed point theorems in a subclass of probabilistic metric spaces. Subsequently a number of fixed point theorems were proved in probabilistic metric and Menger spaces (for an extensive bibliography, refer to Singh-Mishra-Pant [10]). Hicks [2] observed that fixed point theorems for certain contractive type mappings on a Menger space with «minimum norm» may be obtained from corresponding theorems in metric spaces. In this paper we establish a fixed point theorem for a family of mappings in Menger spaces and extend this result to uniform spaces.. These results generalize and extend, among others, the result of Iséki [3], Rhoades [5], Singh [8], Singh-Mishra [9] and Tarafdar [11].

1 - Preliminaries

A probabilistic metric space (PM-space) is an ordered pair $(X, \mathscr{F})$ where X is a nonempty set and $\mathscr{F}$ is a mapping from $X \times X$ to L, the collection of all distribution functions. The value of $\mathscr{F}$ at $(u, v) \in X \times X$ is denoted by $F_{u,v}$ and

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the functions $F_{u,v}$ are assumed to satisfy the following conditions:

(a) $F_{u,v}(x)=1$ for all $x>0$ iff $u=v$.

(b) $F_{u,v}(0)=0$. (c) $F_{u,v}=F_{v,u}$.

(d) If $F_{u,v}(x)=1$ and $F_{v,w}(y)=1$, then $F_{u,w}(x+y)=1$.

A Menger space is a triplet $(X, \mathscr{F}, t)$ where $(X, \mathscr{F})$ is a PM-space and t-norm t [6] is such that the inequality

(e) $F_{u,w}(x+y)\geqslant t\{F_{u,v}(x), F_{v,w}(y)\}$ holds for all u, v, w in X and all $x\geqslant 0$, $y\geqslant 0$.

Theorem 1. *Let $(X, \mathscr{F}, t)$ be a Menger space with t-norm, t satisfying $t(x, x)\geqslant x$, $x\in[0, 1]$ and $\{S_i\}_{i\in N}: X\to X$. If there exist a constant $k\in(0, 1)$ and a mapping $T: X\to X$ such that $S_i(X)\subset T(X)$, $i\in N$ and for every $u, v\in X$, $i, j\in N$, $i\neq j$,*

(1.1)
$$F_{S_iu,S_jv}(kx)$$
$$\geqslant \min\{F_{Tu,Tv}(x), F_{S_iu,Tu}(x), F_{S_jv,Tv}(x), F_{S_iu,Tv}(2x), F_{S_jv,Tu}(2x)\} \quad \textit{for all } x>0$$

(1.2) *$T(X)$ is a complete subspace of X*

(1.3) *each S_i commutes with T*

then for each $i\in N$, T and the family $\{S_i\}$ have a unique common fixed point.

Proof. Pick $u_0\in X$. Since $S_i(X)\subset T(X)$, we can construct a sequence $\{u_n\}$ in X such that

(1.4) $$Tu_n=S_nu_{n-1} \qquad (n=1, 2, \ldots).$$

By (1.1) and (1.4)

$$F_{Tu_1,Tu_2}(kx)=F_{S_1u_0,S_2u_1}(kx)$$
$$\geqslant \min\{F_{Tu_0,Tu_1}(x), F_{Tu_1,Tu_0}(x), F_{Tu_2,Tu_1}(x), F_{Tu_1,Tu_1}(2x), F_{Tu_2,Tu_0}(2x)\}.$$

Since by (e)

$$F_{Tu_2,Tu_0}(2x) \geqslant \min\{F_{Tu_2,Tu_1}(x),\ F_{Tu_1,Tu_0}(x)\}$$

we get

$$F_{Tu_1,Tu_2}(kx) \geqslant F_{Tu_0,Tu_1}(x).$$

Similarly

$$F_{Tu_2,Tu_3}(kx) \geqslant F_{Tu_1,Tu_2}(x).$$

Thus, in general

$$F_{Tu_n,Tu_{n+1}}(kx) \geqslant F_{Tu_{n-1},Tu_n}(x).$$

So, in view of Lemma [**10**], $\{Tu_n\}$ is a Cauchy sequence and has a limit in $T(X)$. Call it p. Then there exists a point z in X such that $Tz = p$.

For $\varepsilon > 0$, $\lambda > 0$, let $U_{S_i z}(\varepsilon, \lambda)$ be a neighbourhood of $S_i z$. Since $Tu_n \to Tz$ there exists an integer $N(\varepsilon, \lambda)$ such that

$$(1.5) \qquad m \geqslant N \text{ implies } F_{Tu_m,Tz}\left(\frac{1-k}{2k}\varepsilon\right) > 1-\lambda \text{ and } F_{Tu_{m+1},Tz}\left(\frac{1-k}{2k}\varepsilon\right) > 1-\lambda.$$

Now by (1.1)

$$F_{Tu_{m+1},S_i z}(\varepsilon)$$

$$\geqslant \min\{F_{Tu_m,Tz}(\varepsilon/k),\ F_{Tu_{m+1},Tu_m}(\varepsilon/k),\ F_{S_i z,Tz}(\varepsilon/k),\ F_{Tu_{m+1},Tz}(2\varepsilon/k),\ F_{S_i z,Tu_m}(2\varepsilon/k)\}$$

$$\geqslant \min\{F_{Tu_m,Tz}\left(\frac{1-k}{2k}\varepsilon\right),\ F_{Tu_{m+1},Tz}\left(\frac{1-k}{2k}\varepsilon\right),\ F_{Tz,Tu_m}\left(\frac{1+k}{2k}\varepsilon\right),$$

$$F_{S_i z,Tu_{m+1}}\left(\frac{1+k}{2k}\varepsilon\right),\ F_{Tu_{m+1},Tz}\left(\frac{1-k}{2k}\varepsilon\right)\}$$

$$= \min\{F_{Tu_{m+1},Tz}\left(\frac{1-k}{2k}\varepsilon\right),\ F_{Tu_m,Tz}\left(\frac{1-k}{2k}\varepsilon\right)\} > 1-\lambda \qquad \text{for all } m \geqslant N.$$

Consequently $Tz = p = S_i z$, and this is true for each $i \in N$.

Nothing that $Tp = T(S_i z) = S_i(Tz) = S_i p$, we get by (1.1)

$$F_{p,Tp}(kx) > F_{p,Tp}(x), \quad x > 0 .$$

This proves $Tp = p$. Moreover

$$S_i p = S_i Tz = TS_i z = Tp = p .$$

The uniqueness of the common fixed point follows from (1.1).

Remark 1. The above result is an extension to Menger spaces of, among others, the results of Iséki [3], Rhoades [5], Singh [8] and Tiwari-Singh [12].

2 - Extension to uniform spaces

Let $D = \{d_\alpha\}$ be a nonempty collection of pseudometrics on X. The uniformity generated by D is obtained by taking as a subbase all sets of the form $U_{\alpha,\varepsilon} = \{(u, v) \in X \times X: d_\alpha(u, v) < \varepsilon\}$, wherein $d_\alpha \in D$ and $\varepsilon > 0$. In fact, the topology generated by this uniformity has all d_α-spheres as a subbase. For details refer to Kelley [4]. Cain, Jr. and Kasriel [1] have shown that a collection of pseudometrics $\{d_\alpha\}$ can be defined which generates the usual structure for Menger spaces. Hence the following result is a direct consequence of Theorem 1.

Theorem 2. *Let X be a Hausdorff space and $\{S_i\}_{i \in N}: X \to X$. If for every $d_\alpha \in D$ there exist a constant $k_\alpha \in (0, 1)$ and a mapping $T: X \to X$ such that $S_i(X) \subset T(X)$, $i \in N$, for each $u, v \in X$, $i, j \in N$ $(i \neq j)$,*

(2.1) $$d_\alpha(S_i u, S_j v)$$

$$\leq k_\alpha \max\{d_\alpha(Tu, Tv),\ d_\alpha(S_i u, Tu),\ d_\alpha(S_j v, Tv),\ \tfrac{1}{2} d_\alpha(S_i u, Tv),\ \tfrac{1}{2} d_\alpha(S_j v, Tu)\}$$

(2.2) *$T(X)$ is a sequentially complete subspace of X*

(2.3) *each S_i commutes with T*

then T and the family $\{S_i\}$ have a unique common fixed point.

Remark 2. The above result includes a number of fixed point theorems in uniformizable spaces. For instance, the results of Singh-Mishra [**9**] and Tarafdar [**11**] may be obtained with suitable choice of $\{S_i\}$ and T in the above theorem.

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Abstract

In this paper we establish a common fixed point theorem for a family of mappings in Menger spaces with an extension of it to uniform spaces. Our results extend and unify a number of fixed point theorems in metric, Menger and uniformizable spaces.

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