Riv. Mat. Univ. Parma (4) **14** (1988), 107-112

HÜSEYIN BOR (*)

A relation between two summability methods (**)

1 - Introduction

Let $\sum a_n$ be a given infinite series with partial sums s_n and $r_n = na_n$. By u_n^α and t_n^α we denote the n-th Cesàro means of order α ($\alpha > -1$) of the sequences (s_n) and (r_n), respectively. The series $\sum a_n$ is said to be summable $|C, \alpha; \gamma|_k$ ($k \geq 1$, $\gamma \geq 0$) if (see [2])

(1.1) $$\sum_{n=1}^{\infty} n^{\gamma k+k-1} |u_n^\alpha - u_{n-1}^\alpha|^k < \infty .$$

Since $t_n^\alpha = n\ (u_n^\alpha - u_{n-1}^\alpha)$ (see [3]), condition (1.1) can also be written as

(1.2) $$\sum_{n=1}^{\infty} \frac{1}{n^{1-\gamma k}} |t_n^\alpha|^k < \infty .$$

Let (p_n) be a sequence of positive real constants such that

(1.3) $$P_n = \sum_{v=0}^{n} p_v \to \infty \quad \text{as } n \to \infty \qquad (P_{-i} = p_{-i} = 0,\ i \geq 1) .$$

The sequence-to-sequence transformation

(1.4) $$T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v$$

(*) Indirizzo: Department of Mathematics, Erciyes University, TR-Kayseri-38039.
(**) Ricevuto: 5-II-1988.

defines the sequence (T_n) of $(\bar{N},\ p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n). The series $\sum a_n$ is said to be summable $|\bar{N},\ p_n|_k$ $(k \geqslant 1)$ if (see [1]$_1$)

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |T_n - T_{n-1}|^k < \infty . \tag{1.5}$$

The series $\sum a_n$ is said to be summable $|\bar{N},\ p_n;\ \gamma|_k$ $(k \geqslant 1,\ \gamma \geqslant 0)$ if (see [1]$_3$)

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\gamma k+k-1} |T_n - T_{n-1}|^k < \infty . \tag{1.6}$$

In the special case when $p_n = 1$ for all values of n (resp. $\gamma = 0$) $|\bar{N},\ p_n;\ \gamma|_k$ summability is the same as $|C,\ 1;\ \gamma|_k$ (resp. $|\bar{N},\ p_n|_k$) summability.

2 – Quite recently Bor [1]$_2$ has established a relation between the $|\bar{N},\ p_n|_k$ and $|C,\ 1|_k$ summability methods. He proved the following theorem.

Theorem A. Let (p_n) be a sequence of positive real constant such that as $n \to \infty$

$$\text{(i)}\quad np_n = O(P_n) \qquad\qquad \text{(ii)}\quad P_n = O(np_n) . \tag{2.1}$$

If $\sum a_n$ is summable $|\bar{N},\ p_n|_k$, then it is also summable $|C,\ 1|_k$, $k \geqslant 1$.

Notice that, to see the hypotheses (i) and (ii) in the Theorem A are satisfied by at least one $p_n \neq 1$, it is sufficient to take $p_n = n$ for all $n \in N$.

3 – The aim of this paper is to establish a relation between the $|\bar{N},\ p_n;\ \gamma|_k$ and $|C,\ 1;\ \gamma|_k$ summability methods. In particular, we have pointed out that $|C,\ 1;\ \gamma|_k$ summability method can be obtained from $|\bar{N},\ p_n;\ \gamma|_k$ summability method by taking $p_n = 1$ for all $n \in N$. However, it should be remarked that one can find a sequence (p_n) so that the methods $|\bar{N},\ p_n;\ \gamma|_k$ and $|C,\ 1;\ \gamma|_k$ are different from each other, where $k \geqslant 1$ and $\gamma \geqslant 0$. In this paper we shall prove the following theorem which includes the Theorem A for $\gamma = 0$. Our theorem is as follows.

Theorem. *Let $k \geqslant 1$, $\gamma \geqslant 0$ and $1-\gamma k>0$. Let (p_n) be a sequence of positive real constants such that satisfy the condition* (2.1). *If $\sum a_n$ is summable $|\bar{N}, p_n; \gamma|_k$, then it is also summable $|C, 1; \gamma|_k$.*

4 – Proof of the Theorem. We have

$$T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v = \frac{1}{P_n} \sum_{v=0}^{n} (P_n - P_{v-1}) a_v . \tag{4.1}$$

The series $\sum a_n$ is summable $|\bar{N}, p_n; \gamma|_k$ so that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\gamma k+k-1} |\Delta T_{n-1}|^k < \infty \tag{4.2}$$

where

$$\Delta T_{n-1} = -\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v .$$

So
$$P_{n-1} a_n = -\frac{P_n P_{n-1}}{p_n} \Delta T_{n-1} + \frac{P_{n-1} P_{n-2}}{p_{n-1}} \Delta T_{n-2}$$

that is

$$a_n = -\frac{P_n}{p_n} \Delta T_{n-1} + \frac{P_{n-2}}{p_{n-1}} \Delta T_{n-2} . \tag{4.3}$$

It is easily verified that this holds also when $n=1$ (since in this case $P_{n-2}=0$). We denote by t_n that the n-th $(C, 1)$ mean of the sequence (na_n). That is

$$t_n = \frac{1}{n+1} \sum_{v=1}^{n} v a_v . \tag{4.4}$$

Since, $a_v = \frac{P_v}{p_v} \Delta T_{v-1} + \frac{P_{v-2}}{p_{v-1}} \Delta T_{v-2}$, by (4.3) we have

$$t_n = \frac{1}{n+1} \left\{ \sum_{v=1}^{n-1} \frac{1}{p_v} \Delta T_{v-1} [-vP_v + (v+1) P_{v-1}] \right\} - \frac{nP_n}{(n+1) p_n} \Delta T_{n-1} .$$

Also, since $-vP_v+(v+1)P_{v-1}=P_v-(v+1)p_v$, we have

$$t_n=\frac{1}{n+1}\sum_{v=1}^{n-1}\frac{P_v}{p_v}\Delta T_{v-1}-\frac{1}{n+1}\sum_{v=1}^{n-1}(v+1)\Delta T_{v-1}-\frac{nP_n}{(n+1)p_n}\Delta T_{n-1}$$

$$=t_{n,1}+t_{n,2}+t_{n,3}.$$

To prove the theorem, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty}\frac{1}{n^{1-\gamma k}}|t_{n,r}|^k<\infty \qquad \text{for } r=1,\ 2,\ 3.$$

Now, applying Hölder's inequality, we have

$$\sum_{n=2}^{m+1}\frac{1}{n^{1-\gamma k}}|t_{n,1}|^k$$

$$=\sum_{n=2}^{m+1}\frac{1}{n^{1-\gamma k}}\left|\frac{1}{n+1}\sum_{v=1}^{n-1}\frac{P_v}{p_v}\Delta T_{v-1}\right|^k\leqslant\sum_{n=2}^{m+1}\frac{1}{n^{k-\gamma k+1}}\{\sum_{v=1}^{n-1}\frac{P_v}{p_v}|\Delta T_{v-1}|\}^k$$

$$\leqslant\sum_{n=2}^{m+1}\frac{1}{n^{2-\gamma k}}\sum_{v=1}^{n-1}(\frac{P_v}{p_v})^k|\Delta T_{v-1}|^k\{\frac{1}{n}\sum_{v=1}^{n-1}1\}^{k-1}$$

$$\leqslant O(1)\sum_{v=1}^{m}(\frac{P_v}{p_v})^k|\Delta T_{v-1}|^k\sum_{n=v+1}^{m+1}\frac{1}{n^{2-\gamma k}}=O(1)\sum_{v=1}^{m}(\frac{P_v}{p_v})^k|\Delta T_{v-1}|^k\int_v^{\infty}x^{\gamma k-2}\,\mathrm{d}x$$

$$\leqslant O(1)\sum_{v=1}^{m}(\frac{P_v}{p_v})^k v^{\gamma k}\frac{1}{v}|\Delta T_{v-1}|^k.$$

Since, $v^{\gamma k}=O(\frac{P_v}{p_v})^{\gamma k}$ by (2.1.i) and $\frac{1}{v}=O(\frac{p_v}{P_v})$ by (2.1.ii), we have

$$\sum_{n=2}^{m+1}\frac{1}{n^{1-\gamma k}}|t_{n,1}|^k=O(1)\sum_{v=1}^{m}(\frac{P_v}{p_v})^{\gamma k+k-1}|\Delta T_{v-1}|^k=O(1)$$

as $m\to\infty$ by (4.2).

Again, as in $t_{n,1}$ we have

$$\sum_{n=2}^{m+1} \frac{1}{n^{1-\gamma k}} |t_{n,2}|^k$$

$$= \sum_{n=2}^{m+1} \frac{1}{n^{1-\gamma k}} \left| \frac{1}{n+1} \sum_{v=1}^{n-1} \frac{v+1}{v} v \Delta T_{v-1} \right|^k = O(1) \sum_{n=2}^{m+1} \frac{1}{n^{1-\gamma k+k}} \{ \sum_{v=1}^{n-1} v |\Delta T_{v-1}| \}^k$$

$$= O(1) \sum_{n=2}^{m+1} \frac{1}{n^{2-\gamma k}} \sum_{v=1}^{n-1} v^k |\Delta T_{v-1}|^k \cdot \{ \frac{1}{n} \sum_{v=1}^{n-1} 1 \}^{k-1}$$

$$= O(1) \sum_{v=1}^{m} v^{\gamma k+k} \frac{1}{v} |\Delta T_{v-1}|^k = O(1) \sum_{v=1}^{m} (\frac{P_v}{p_v})^{\gamma k+k-1} |\Delta T_{v-1}|^k = O(1)$$

as $m \to \infty$ by (4.2).

Finally, we have

$$\sum_{n=1}^{m} \frac{1}{n^{1-\gamma k}} |t_{n,3}|^k$$

$$= \sum_{n=1}^{m} \frac{1}{n^{1-\gamma k}} \left| \frac{nP_n}{(n+1)p_n} \Delta T_{n-1} \right|^k = O(1) \sum_{n=1}^{m} (\frac{P_n}{p_n})^k n^{\gamma k} \frac{1}{n} |\Delta T_{n-1}|^k$$

As in $t_{n,1}$, we have

$$\sum_{n=1}^{m} \frac{1}{n^{1-\gamma k}} |t_{n,3}|^k = O(1) \sum_{n=1}^{m} (\frac{P_n}{p_n})^{\gamma k+k-1} |\Delta T_{n-1}|^k = O(1)$$

as $m \to \infty$ by (4.2). Therefore, we get

$$\sum_{n=1}^{m} \frac{1}{n^{1-\gamma k}} |t_{n,r}|^k = O(1) \quad \text{as } m \to \infty \quad \text{for } r = 1,\ 2,\ 3.$$

This completes the proof of the theorem.

References

[1] H. BOR: [•]$_1$ *On $|\bar{N}, p_n|_k$ summability factors of infinite series*, Tamkang J. Math. (1) **16** (1985), 13-20; [•]$_2$ *A note on two summability methods*, Proc. Amer. Math. Soc. **98** (1986), 81-84; [•]$_3$ *A note on $|\bar{N}, q_n|_k$ summability factors*, Pure Appl. Math. Sci. **XXIV** (1986), 17-23.

[2] T. M. FLETT, *Some more theorems concerning the absolute summability of Fourier series*, Proc. London Math. Soc. 8 (1958), 357-385.

[3] E. KOGBETLIANTZ, *Sur les séries absolument sommables par la méthode des moyannes arithmétiques*, Bull. Sci. Math. **49** (1925), 234-256.

Summary

In this paper a relation between the $|\bar{N}, p_n; \gamma|_k$ *and* $|C, 1; \gamma|_k$ *summability methods, which generalizes the result of Bor* $[\mathbf{1}]_2$, *has been established.*

***