Riv. Mat. Univ. Parma (4) **14** (1988), 113-119

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On the oscillatory properties of the solutions of non-linear neutral functional differential equations of second order (**)

1 - Introduction

In the present paper sufficient conditions have been obtained for oscillation or tending to zero of all bounded solutions of equations of the form

$$[A(x_t)]'' + \rho(t)B(x_t) = 0 \tag{1}$$

where $x_t(\theta) = x(t+\theta)$, $\theta \in [-\tau, 0]$, $\tau = \text{const} > 0$ and the functionals $A, B : C[-\tau, 0) \to \mathbb{R}$ are monotonic.

The oscillatory properties of linear and non-linear ordinary differential and functional differential equations have been an object of investigation by many authors [2]... [5], [8], [10]. The neutral equations of second order have numerous applications (see for instance [1], [6]) but their oscillatory and asymptotic properties are studied comparatively little. Some results in this direction for the case when the function $\rho(t)$ is non-negative have been obtained in [9], [11].

2 - Preliminary notes

Def. 1. We shall say that the function $\varphi : I_\varphi \to \mathbb{R}$ ($I_\varphi = [t_\varphi, \infty)$, $t_\varphi \in \mathbb{R}$, $\infty = +\infty$) is *oscillating* if $\sup\{t | \varphi(t) = 0\} = \infty$ and $\sup\{t | \varphi(t) \neq 0\} = \infty$.

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(**) Ricevuto: 16-II-1988.

Def. 2. A function $x: I_x \to \mathbb{R}$ will be called *a solution of equation* (1) if $x \in C(I_x)$, $A(x_t) \in C^2(I_x + \tau)$ and satisfies equation (1) for $t \in I_x + \tau$.

By $\Omega^{\alpha,\beta}$ $(0 < \beta \leqslant \alpha)$ we shall denote the set of all continuous functionals $A: C[-\tau, 0] \to \mathbb{R}$ which satisfy the following conditions:

A1. For any function $\varphi \in C[-\tau, 0]$ with the property $\varphi(t) \neq 0$, $t \in [-\tau, 0]$, the following equality holds

$$\operatorname{sgn} A(\varphi) = \operatorname{sgn} \varphi(0).$$

A2. For any $\varepsilon > 0$ there exists $\delta > 0$ such that for any function $\varphi \in C[-\tau, \tau]$ with the property $\min\limits_{[-\tau, \tau]} |\varphi(t)| > 0$ the inequality $\max\limits_{[0, \tau]} |A(\varphi_t)| < \delta$ implies the inequality $|\varphi(0)| < \varepsilon$.

A3. For all constants b_1, b_2, $0 < b_1 \leqslant b_2$, and any function $\varphi \in C[-\tau, \alpha]$ with the property $\min\limits_{[-\tau, \alpha]} \varphi(t) > 0$ for which the inequality $b_1 \leqslant |A(\varphi_t)| \leqslant b_2$, $t \in [-\tau, \alpha]$, holds there exists a measurable set $Q \subseteq [-\tau, \alpha]$ and a constant $b_3 > 0$ such that $\mu(Q) \geqslant \beta$ (μ is the Lebesgue measure), $|\varphi(t)| \geqslant b_3$ for $t \in Q$ and the following equality holds

$$\operatorname{sgn} \varphi(t)|_Q = \operatorname{sgn} A(\varphi_t)|_{[0, \alpha]}.$$

Example 1. It is immediately verified that for any α and correspondingly chosen β the functional A defined by the equality $A(\varphi) = \sum\limits_{i=1}^{n} a_i \varphi(-\tau_i)$, $n \geqslant 1$, $a_i > 0$, $0 \leqslant \tau_i \leqslant \tau$, $(i = \overline{1, n})$ belongs to the set $\Omega^{\alpha,\beta}$.

For the function $\rho: I_p \to \mathbb{R}$ we introduce the notation

$$E_\rho^+ = \{z \in I_\rho | \rho(t) \geqslant 0\} \qquad E_\rho^- = \{t \in I_\rho | \rho(t) \leqslant 0\}.$$

By ρ^γ, $\gamma > 0$, we shall denote the set of continuous functions $\rho: I_p \to \mathbb{R}$ satisfying the following property:

P1. There exists a number $\varepsilon > 0$ and a point $t_0 \in I_p$ such that for any $t \geqslant t_0$ for which $\rho(t) > 0$ one can find an interval $[t', t''] \subset I_p$ with length $t'' - t' \geqslant \gamma + \varepsilon$ with the property $t \in [t', t''] \subset E_{\varphi}^{+}$ (i.e. the intervals in which the function is positive should be large enough).

By Λ we shall denote the set of coontinuous functionals $B : C[-\tau, 0] \to \mathbb{R}$ satisfying the following properties:

B1. For any element $\varphi \in C[-\tau, 0]$ with the property $\min_{[-\tau, 0]} |\varphi(t)| > 0$ the following equality holds

$$\operatorname{sgn} B(\varphi) = \operatorname{sgn} \varphi(0).$$

B2. For any $\varepsilon > 0$ there exists $\delta > 0$ such that for any element $\varphi \in C[-\tau, 0]$ with the property $\min_{[-\tau, 0]} |\varphi(t)| > 0$ for which the inequality $|\varphi(0)| \geqslant \varepsilon$ holds, the inequality $B(\varphi) \geqslant \delta$ holds as well.

B3. $B(s \cdot 1(\cdot))$ is a non-decreasing function for $s \in \mathbb{R}$, $1(t) \equiv 1$ and the following relation holds

$$\int_0^1 \left[\frac{1}{B(s \cdot 1(\cdot))} + \frac{1}{|B(-s \cdot 1(\cdot))|}\right] ds < \infty .$$

Remark 1. We shall note that from condition B3 it follows that no functional $B \in \Lambda$ can be linear.

Lemma 1. *Let the function $h : [a, b] \to [0, \infty)$ be absolutely continuous, $\varphi \in C^2[a, C]$ and let the function $f \in C[\min \varphi, \max \varphi]$ be non-increasing.*
Then the following inequality holds

$$\int_a^b h(t)\, \varphi''(t)\, f(\varphi(t))\, dt$$

$$\geqslant h(b)\, \varphi'(b)\, f(\varphi(b)) - h(a)\, \varphi'(a)\, f(\varphi(a)) - \int_a^b h'(t)\, \varphi'(t)\, f(\varphi(t))\, dt .$$

Proof. If f is of class C^1, then the assertion of the lemma is proved by an integration by parts and in the case when f is of class C by means of uniform approximation of f by non-increasing functions of class C^1.

Theorem 2. *Let for equation* (1) *numbers α, β $(0 < \beta \leqslant \alpha)$ exist such that* *the following conditions be fulfilled:*

1. $A \in \Omega^{\alpha,\beta}$ 2. $\rho \in \rho^{\alpha+\tau}$ 3. $B \in \Lambda$.

4. *For any constant* $a > 0$ *the following relation holds*

$$\sup_{\{\varphi \in C[-\tau,\, \tau] | 0 < |\varphi(t)| \leq a\}} \frac{B(\varphi)}{B(A(\varphi) \cdot 1(\cdot))} < \infty .$$

5. *There exists a locally absolutely continuous function* $h : I_p \to (0,\ \infty)$ *with the properties* $\operatorname*{Var}_{[t_\rho,\, t]} h = 0(t)$ *for* $t \to \infty$, $\operatorname*{Var}_{[t_\rho,\, \infty)} h' < \infty$, *for which the following relation holds*

$$\int_{E_\rho^-} h(t) |\rho(t)|\, \mathrm{d}t < \infty . \tag{2}$$

6. *There exists a number* $\varepsilon > 0$ *for which the following inequality is satisfied*

$$\limsup_{t \to \infty} \mu \{ s \in [t,\ t + \alpha + \tau] | h(s) \rho(s) \leq \varepsilon \} < \beta .$$

Then each bounded solution of equation (1) *either oscillates or tends to zero for* $t \to \infty$.

Proof. Let $x : I_x \to \mathbb{R}$ be a solution of equation (1) which is not identically equal to zero for sufficiently large values of t. Without loss of generality we can assume that $x(t) > 0$ for $t \in I_x$. Multiplying both sides of equation (1) by the expression $\dfrac{h(t)}{B(A(x_t) \cdot 1(\cdot))}$ and integrating from $t_1 = t_x + \tau$ to $t > t_1$ we obtain the equality

$$\int_{t_1}^{t} \frac{[A(x_s)]'' h(s)\, \mathrm{d}s}{B(A(x_s) \cdot 1(\cdot))} + \int_{t_1}^{t} h(s) \rho(s) \frac{B(x_s)}{B(A(x_s) \cdot 1(\cdot))} \mathrm{d}s = 0 .$$

Applying to the first integral Lemma 1 and integrating once more from t_1 to $t > t_1$ we obtain the inequality

$$\int_{t_1}^{t} \frac{h(s)[A(x_s)]'}{B(A(x_s) \cdot 1(\cdot))} \mathrm{d}s - \frac{h(t_1)[A(x_t)]'|_{t=t_1}}{B(A(x_{t_1}) \cdot 1(\cdot))} (t - t_1) \tag{3}$$

$$- \int_{t_1}^{t} \Big(\int_{t_1}^{s} \frac{h'(y)[A(x_y)]'}{B(A(x_y) \cdot 1(\cdot))} \mathrm{d}y \Big) \mathrm{d}s + \int_{t_1}^{t} \Big(\int_{t_1}^{s} h(y) \rho(y) \frac{B(x_y)}{B(A(x_y) \cdot 1(\cdot))} \mathrm{d}y \Big) \mathrm{d}s \leq 0 .$$

Taking into account the properties of the function $h(t)$ and setting $\phi(t) = \int_0^t \frac{ds}{B(s \cdot 1(\cdot))}$ we obtain for $t \to \infty$ the following relations

(4)
$$\int_{t_1}^{t} \frac{h(s)[A(x_s)]'\,ds}{B(A(x_s)\cdot 1(\cdot))} = \int_{t_1}^{t} h(s)\,d\phi(A(x_s))$$
$$= h(t)\,\phi(A(x_t)) - h(t_1)\,\phi(A(x_{t_1})) - \int_{t_1}^{t} \phi(A(x_s))\,dh(s) = 0(t).$$
$$\int_{t_1}^{t} \frac{h'(s)[A(x_s)]'\,ds}{B(A(x_s)\cdot 1(\cdot))} = \int_{t_1}^{t} h'(s)\,d\phi(A(x_s))$$
$$= h'(t)\,\phi(A(x_t)) - h'(t_1)\,\phi(A(x_{t_1})) - \int_{t_1}^{t} \phi(A(x_s))\,dh'(s) = 0(1).$$

From inequality (3), in view of relations (2), (4) and condition 4 of Theorem 2, we obtain for $t \to \infty$ the relation

(5)
$$\int_{t_1}^{t} \Big(\int_{[t_1,\, s]\cap E_\rho^+} h(y)\,\rho(y) \frac{B(x_y)}{B(A(x_y)\cdot 1(\cdot))}\,dy \Big)\,ds = 0(t).$$

We shall prove that the following relation holds

(6)
$$\int_{[t_1,\, \infty)\cap E_\rho^+} h(t)\,\rho(t) \frac{B(x_t)}{B(A(x_t)\cdot 1(\cdot))}\,dt = \infty$$

which obviously contradicts relation (5).

From condition A2 it follows that $\lim_{t\to\infty} \sup A(x_t) > 0$, so let us set $C := \lim_{t\to\infty} \sup A(x_t)$. On the other hand, from equation (1) it follows that the function $A(x.)$ is concave (convex) in any interval belonging to $\{I_x + \tau\} \cap E_\rho^+$ ($\{I_x + \tau\} \cap E_\rho^-$). In view of condition 6 of Theorem 2 we conclude that $\sup E_\rho^+ = \infty$, hence there exists a sequence $\{t_i\} \subset E_\rho^+$ with the property $\lim_{i\to\infty} (t_{i+1} - t_i) = \infty$ such that $\lim_{i\to\infty} A(x_{t_i}) = C$. From condition P1 it follows that there exists a sequence of disjoint intervals $\{l_i\}$, $t_i \in l_i$, with length $\alpha + \tau$ such that the inequality $\inf_i \min_{t\in Q_i} A(x_t) > 0$ holds.

Then by condition A3 there exists measurable sets $Q_i \subset l_i$ with the property $\mu(Q_i) \geqslant \beta$ $(i = 1, 2, \ldots)$, such that the inequality $\inf_i \min_{t\in Q_i} x(t) > 0$ holds. From last inequality and condition B2 it follows that $\inf_i \inf_{t \in Q_i} B(x_t) > 0$, hence the following inequality holds

$$\inf_i \inf_{t \in Q_i} \frac{B(x_t)}{B(A(x_t) \cdot 1(\cdot))} > 0 . \tag{7}$$

From condition 6 of Theorem 2 it follows that there exist sets $Q_i' \subseteq Q_i$ for which $\lim_{i \to \infty} \inf \mu(Q_i') > 0$ and the inequality

$$\lim_{i \to \infty} \inf \inf_{t \in Q_i'} h(t) \rho(t) > 0$$

holds. Inequalities (7) and (8) immediately imply relation (6).

Remark 2. If, moreover, it is given that the function $\rho(t) \geqslant 0$, then each bounded solution which for sufficiently large values of t is not identically zero oscillates. In this case, if $x(t) \geqslant 0$ for $t \geqslant t_x$, then the function $A(x)$ for $t \geqslant t_x$ is concave, hence $x(t)$ may tend to zero for $t \to \infty$ only if it is identically zero for $t > t_x$.

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Summary

In the present paper sufficient conditions have been obtained for oscillation or tending to zero of all bounded solutions of equations of the form $[A(x_t)]'' + \rho(t)B(x_t) = 0$, *where* $x_t(\theta) = x(t+\theta)$, $\theta \in [-\tau, 0]$, $\tau = \text{const} > 0$, $\rho: \mathrm{R} \to \mathrm{R}$ *and the functionals* $A, B: C[-\tau, 0] \to \mathrm{R}$ *are monotonic.*

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