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Sums of near-rings (**)

1 - Introduction

In this paper sums of near-rings are defined and studied in order to characterize near-rings whose additive group is the direct sum or the semidirect sum of additive groups.

2 - Preliminaries

A left near-ring is indicated by N; for the definitions and the fundamental notations we refer to $[3]_2$ without express recall. In particular the additive group of N is indicated by N^+; if A is a substructure of N, $\mathscr{F}(A) \subseteq \text{AUT}(N^+)$ denotes the subset of automorphisms of N^+ which transforms A in itself. If f, g are functions from S to T and $H \subseteq T$, we write $f =_H g$ for $f(x) - g(x) \in H$ $\forall x \in S$.

Moreover $\gamma_a : x \to ax$ $\forall x \in N$ is a left translation of N determined by a; $A_d(N) = \{x \in N / Nx = 0\}$ ($A_s(N) = \{x \in N / xN = 0\}$) is right (left) annihilator of N. In the following we call a near-ring $N = N_0 + N_c$ with $N_0 \neq \{0\} \neq N_c$ *mixed*.

3 - Semidirect sum of groups

We recall (see [4]) that if A and B are additive groups and $\varphi : B^+ \to \text{AUT}(A^+)$ is a homomorphism with $\varphi(b) = \varphi_b$, the structure $[A \times B, +]$ with $\langle a, b \rangle + \langle a', b' \rangle = \langle a + \varphi_b(a'), b + b' \rangle$ is an additive group, called the *semidi-*

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rect sum of A and B with homomorphism φ and we will indicate it with $A +_\varphi B$. If $A^0 = \{\langle a, 0\rangle / a \in A\}$ and ${}^0B = \{\langle 0, b\rangle / b \in B\}$ we know that:

(a) A^0 and ${}^0B$ are subgroups of $A +_\varphi B$; (b) $A +_\varphi B = A^0 + {}^0B$;

(c) $A^0 \cap {}^0B = \{\langle 0, 0\rangle\}$; (d) $A^0 \cong A$ and $A + B/A^0 \cong B$.

Proposition 1. *Let* $G = A +_\varphi B$, $\bar{A} \subseteq A$ *and* $\bar{B} \subseteq B$, *then:* $[\bar{N} = \bar{A} \times \bar{B}, +]$ *is a subgroup of* G *iff* $\bar{A}$ *and* $\bar{B}$ *are subgroups of* A *and* B *respectively and* $\varphi(\bar{B}) \subseteq \mathscr{F}(\bar{A})$; $\bar{N}$ *is a normal subgroup of* G *iff* $\bar{A}$ *and* $\bar{B}$ *are normal subgroups of* A *and* B *respectively,* $\varphi(B) \subseteq \mathscr{F}(\bar{A})$ *and* $\varphi(\bar{b}) = {}_{\bar{A}}\mathrm{id} \quad \forall \bar{b} \in \bar{B}$.

Proof. Easy verification.

Corollary 1. *Let* $G = A +_\varphi B$, *then* ${}^0B$ *is a normal subgroup of* G *iff* $\ker \varphi = B$.

Corollary 2. *Let* $G = A +_\varphi B$ *and* $\varphi(B) = \{\mathrm{id}\}$, *then:* G *is a direct sum of* A *and* B, $\bar{N} = [\bar{A} \times \bar{B}, +]$ *is a (normal) subgroup of* G *iff* $\bar{A}$ *and* $\bar{B}$ *are (normal) subgroups of* A *and* B *respectively.*

4 - Near-rings whose additive group is a semidirect or direct sum

Def. 1. Let $N = A +_\varphi B$ be a semidirect sum of A of B with homomorphism φ; we define in N: $\langle a, b\rangle \cdot \langle a', b'\rangle = \langle f_{a,b}(a'), b'\rangle$ where $f_{a,b} = \psi(\langle a, b\rangle)$ and $\psi : A \times B \to \mathrm{END}(A^+)$ is a function for which the following conditions hold:

(1) $f_{a,b} \cdot f_{a',b'} = f_{f_{a,b}(a'),b'}$ (2) $f_{0,0} = \underline{0}$

(3) $f_{a,b} \cdot \varphi_{b'} = \varphi_{b'} \cdot f_{a,b}$.

Proposition 2. *The structure* $N = [A +_\varphi B, \cdot]$ *as in* Def. 1, *is a left near-ring in which* $A^0 = N_0$, ${}^0B = N_c$ *and* N_0 *is a two-sided ideal.*

Proof. Easy verification.

The structure above mentioned will be called *α-sum of* A and B and will be indicated by $A +_\alpha B$.

Theorem 1. *A mixed near-ring N has N_0 as a two-sided ideal iff it is isomorphic to an α-sum of N_0 and N_c.*

Proof. Let N be a mixed near-ring and N_0 a two-sided ideal of N; then N^+ is isomorphic to $N_0 +_\varphi N_c$, where $\varphi: N_c^+ \to \mathrm{AUT}(N_0^+)$ is defined by $[\varphi(n_c)](n_0) = n_c + n_0 - n_c$.

Moreover we consider the function $\psi: N_0 \times N_c \to \mathrm{END}(N_0^+)$ defined by $\psi(\langle n_0, n_c\rangle) = \gamma_n$ where $n = n_0 + n_c$; ψ fulfills the conditions of Def. 1:

(1) $(\gamma_n \cdot \gamma_{n'})(\bar{n}_0) = (n_0 + n_c)[(n_0' + n_c')\bar{n}_0] = [(n_0 + n_c)n_0' + n_c']\bar{n}_0 = \gamma_{\gamma_n(n'_0)+n'_c}(\bar{n}_0)$

(2) $\gamma_0(n_0) = 0 \quad \forall n_0 \in N_0$

(3)
$$(\gamma_n \cdot \varphi_{\bar{n}_c})(\bar{n}_0) = (n_0 + n_c)(\bar{n}_c + \bar{n}_0 - \bar{n}_c) = \bar{n}_c + (n_0 + n_c)\bar{n}_0 - \bar{n}_c$$
$$= \varphi_{\bar{n}_c}(\gamma_n(\bar{n}_0)) = (\varphi_{\bar{n}_c} \cdot \gamma_n)(\bar{n}_0).$$

We can easily verify that the correspondence h: $n_0 + n_c \to \langle n_0, n_c\rangle$ is an isomorphism from N to $N_0 +_\alpha N$.

The converse follows from Proposition 2.

Obviously the multiplication of N infers a multiplication in A and B if we define $a \cdot a' = \pi_A\ (\langle a, 0\rangle \cdot \langle a', 0\rangle)$ and $b \cdot b' = \pi_B\ (\langle 0, b\rangle \cdot \langle 0, b'\rangle)$ and with respect to such operations A and B are a zero-symmetric and a constant near-ring respectively.

Proposition 3. *Let A be a zero-symmetric near-ring, B a constant near-ring and $N = A +_\alpha B$. The multiplication inferred in A by multiplication of N and multiplication of A coincide iff we define $f_{a,0} = \gamma_a \quad \forall a \in A$.*

Proof. Easy verification.

Proposition 4. *Let $N = A +_\alpha B$; a subset $\bar{N}$ of N is:*

(a) *a subnear-ring of N iff $\bar{N} = \bar{A} \times \bar{B}$ where $\bar{A}$ and $\bar{B}$ are subnear-rings of A and B respectively, $\varphi(\bar{B}) \subseteq \mathscr{F}(\bar{A})$ and $\psi(\bar{N}) \subseteq \mathscr{F}(\bar{A})$;*

(b) *a left ideal of N iff $\bar{N} = \bar{A} \times \bar{B}$ where $\bar{A}$ and $\bar{B}$ are left ideals of A and B respectively, $\varphi(B) \subseteq \mathscr{F}(\bar{A})$, $\varphi(\bar{b}) = {}_{\bar{A}}\mathrm{id}\ \forall \bar{b} \in \bar{B}$, $\psi(N) \subseteq \mathscr{F}(\bar{A})$;*

(c) *a right ideal of N iff $\bar{N} = \bar{A} \times \bar{B}$ where $\bar{A}$ and $\bar{B}$ are right ideals of A and B respectively, $\varphi(B) \subseteq \mathscr{F}(\bar{A})$, $\varphi(\bar{b}) = {}_{\bar{A}}\mathrm{id}\ \forall \bar{b} \in \bar{B}$, $f_{a+\varphi_b(\bar{a}), b+\bar{b}} = {}_{\bar{A}}f_{a,b}\ \forall a \in A$, $\forall \bar{a} \in \bar{A}$, $\forall b \in B$, $\forall \bar{b} \in \bar{B}$.*

Proof. (a) Let $\bar{N}$ be a subnear-ring of N; since $\forall\langle a, b\rangle \in \bar{N}$ $\langle 0, 0\rangle \cdot \langle a, b\rangle = \langle 0, b\rangle \in N$, it follows that $\bar{N} = \bar{A} \times \bar{B}$, where $\bar{A} = \{a \in A / \exists b \in B,$ $\langle a, b\rangle \in \bar{N}\}$ and $\bar{B} = \{b \in B / \exists a \in A, \langle a, b\rangle \in \bar{N}\}$.

In this case, by Proposition 1, we know that $\bar{A}$ and $\bar{B}$ are additive subgroups of A and B respectively and $\varphi(\bar{B}) \subseteq \mathscr{F}(\bar{A})$.

Moreover $\langle a, 0\rangle \cdot \langle a', 0\rangle = \langle f_{a,0}(a'), 0\rangle \in N$ $\forall a, a' \in \bar{A}$, so $f_{a,0}(a') = \gamma_a(a') = a \cdot a' \in \bar{A}$ and $\bar{A}$ is a subnear-ring of A. Lastly $\langle a, b\rangle \cdot \langle a', 0\rangle = \langle f_{a,b}(a'), 0\rangle \in \bar{N}$ $\forall \langle a, b\rangle, \langle a', 0\rangle$ of N, so $\psi(\bar{N}) \subseteq \mathscr{F}(\bar{A})$.

The converse follows easily by Proposition 1 and $\psi(\bar{N}) \subseteq \mathscr{F}(\bar{A})$.

(b) Let $\bar{N}$ be a left ideal of N; obviously, by Proposition 1 and Proposition 4(a), we have $\bar{N} = \bar{A} \times \bar{B}$ where $\bar{A}$ and $\bar{B}$ are subnear-rings of A and B respectively, $\varphi(B) \subseteq \mathscr{F}(\bar{A})$ and $\varphi(\bar{b}) = {}_{\bar{A}}\mathrm{id}$ $\forall \bar{b} \in B$, moreover $\langle a, 0\rangle \cdot \langle a', 0\rangle = \langle f_{a,0}(a'), 0\rangle \in \bar{N}$ $\forall a \in A$ and $\forall a' \in \bar{A}$, so $f_{a,0}(a') = \gamma_a(a') = a \cdot a' \in \bar{A}$ and $\bar{A}$ is a left ideal; obviously $\bar{B}$ is a left ideal of B; lastly $\langle a, b\rangle \cdot \langle a', 0\rangle = \langle f_{a,b}(a'), 0\rangle \in \bar{N}$ $\forall \langle a, b\rangle \in n$ and $\forall \langle a', 0\rangle \in \bar{N}$, so we have $f_{a,b}(a') \in \bar{A}$ and $\psi(N) \subseteq \mathscr{F}(\bar{A})$.

The converse follows easily by Proposition 1 and Proposition 4(a).

(c) Let $\bar{N}$ be a right ideal of N; by Proposition 1 and Proposition 4(a), we have $\bar{N} = \bar{A} \times \bar{B}$ where $\bar{A}$ and $\bar{B}$ are subnear-rings of A and B respectively, $\varphi(B) \subseteq \mathscr{F}(\bar{A})$ and $\varphi(\bar{b}) = {}_{\bar{A}}\mathrm{id}$ $\forall \bar{b} \in \bar{B}$.

Lastly, if $\langle \bar{a}, \bar{b}\rangle \in \bar{N}$ we have $(\langle a, b\rangle + \langle \bar{a}, \bar{b}\rangle) \cdot \langle a', b'\rangle - \langle a, b\rangle \cdot \langle a', b'\rangle = \langle f_{a+\varphi_b(\bar{a}), b+\bar{b}}(a'), b'\rangle + \langle \varphi_{-b'}(-f_{a,b}(a')), -b\rangle = \langle f_{a+\varphi_b(\bar{a}), b+\bar{b}}(a') - f_{a,b}(a'), 0\rangle \in N$, so $f_{a+\varphi_b(\bar{a}), b+\bar{b}} = {}_{\bar{A}} f_{a,b}$.

The converse is as above.

Corollary 1. *Let $N = A +_{\varphi} B$; ${}^0B$ is a left ideal of N iff $B = \ker \varphi$ and it is a right ideal of N iff $B = \mathrm{Ker}\,\varphi$ and $f_{a,b} = f_{a,0}$ $\forall b \in B$.*

Def. 2. Let $N = A \oplus B$ be a direct sum of additive groups A and B; we define in N $\langle a, b\rangle \cdot \langle a', b'\rangle = \langle f_{a,b}(a'), \bar{f}_{a,b}(b')\rangle$ $\forall a, a' \in A$, $\forall b, b' \in B$, where $f_{a,b} = \psi(\langle a, b\rangle)$, $\bar{f}_{a,b} = \bar{\psi}(\langle a, b\rangle)$ and where $\psi: A \times B \to \mathrm{END}(A^+)$, $\bar{\psi}: A \times B \to \mathrm{END}(B^+)$ are functions for which

(1) $f_{a,b} \cdot f_{a',b'} = f_{f_{a,b}(a'), \bar{f}_{a,b}(b')}$ $\quad \bar{f}_{a,b} \cdot \bar{f}_{a',b'} = \bar{f}_{f_{a,b}(a'), \bar{f}_{a,b}(b')}$; $\qquad$ (2) $f_{0,0} = \underline{0} = \bar{f}_{0,0}$.

Proposition 5. *The structure $N = [A \oplus B, \cdot]$ as in* Def. 2, *is a zero-symmetric left near-ring in which A^0 and ${}^0B$ are left ideals.*

Proof. Easy verification.

The structure above mentioned will be called *β-sum* of A and B and will be indicated by $A +_\beta B$.

Theorem 2. *A near-ring N is zero-symmetric with $N^+ = I \oplus J$, where I and J are left ideals with trivial intersection, iff it is isomorphic to a β-sum of I and J.*

Proof. Let N be a zero-symmetric near-ring with $N^+ = I \oplus J$ where I and J are left ideals of N and $I \cap J = \{0\}$.

We consider the functions $\psi : I \times J \to \text{END}(I^+)$ and $\bar{\psi} : I \times J \to \text{END}(J^+)$ defined by: $\psi(\langle i, j\rangle) = \gamma_{i+j}/I^+$ and $\bar{\psi}(\langle i, j\rangle) = \gamma_{i+j}/J^+ = \bar{\gamma}_{i+j}$, that is we consider the left translation γ_{i+j} restricted to I and J respectively.

Such restrictions are obviously endomorphism of I^+ and J^+ respectively, because I and J are left ideals of N; ψ and $\bar{\psi}$ fulfil the conditions of the Def. 2:

$$(1) \quad (\gamma_{i+j} \cdot \gamma_{i'+j'})(i'') = (i+j)(i'+j')\,i'' = [(i+j)\,i' + (i+j)\,j']\,i'' = (\gamma_{i+j}(i') + \gamma_{i+j}(j')))\,i'' = \gamma_{\gamma_{i+j}(i') + \bar{\gamma}_{i+j}(i')}(i'').$$

We can prove the same for $\bar{\gamma}$;

$$(2) \quad \gamma_{0+0}(i) = 0 \cdot i = 0 \quad \forall i \in I \qquad \bar{\gamma}_{0+0}(j) = 0 \cdot j = 0 \quad \forall j \in J.$$

Called $f_{i,j} = \gamma_{i+j}/I^+$ and $\bar{f}_{i,j} = \gamma_{i+j}/J^+$, we can easily verify that the correspondence $h : i + j \to \langle i, j\rangle$ from N to $I +_\beta J$ is an isomorphism.

The converse follows directely from Proposition 5.

The multiplication of N infers a multiplication in A and B if we define $a \cdot a' = \pi_A(\langle a, 0\rangle \cdot \langle a', 0\rangle)$ and $b \cdot b' = \pi_B(\langle 0, b\rangle \cdot \langle 0, b'\rangle)$ and with respect to such operations A and B are zero-symmetric near-rings.

Proposition 6. *Let A and B be zero-symmetric near-rings and $N = A +_\beta B$. The multiplications inferred in A and B by multiplication of N and the multiplications of A and B coincide iff we define $f_{a,0} = \gamma_a$ and $\bar{f}_{0,b} = \bar{\gamma}_b$ $\forall a \in A$, $\forall b \in B$.*

Proof. Easy verification.

Proposition 7. *Let* $N = A +_{\beta} B$; *a subset* $\bar{N} = \bar{A} \times \bar{B}$ *of* N *is*:

(a) *a subnear-ring of* N *iff* $\bar{A}$ *and* $\bar{B}$ *are subnear-rings of* A *and* B *respectively*, $\psi(\bar{N}) \subseteq \mathscr{F}(\bar{A})$, $\bar{\psi}(\bar{N}) \subseteq \mathscr{F}(\bar{B})$;

(b) *a left ideal of* N *iff* $\bar{A}$ *and* $\bar{B}$ *are left ideals of* A *and* B *respectively*, $\psi(N) \subseteq \mathscr{F}(\bar{A})$, $\bar{\psi}(N) \subseteq \mathscr{F}(\bar{B})$;

(c) *a right ideal of* N *iff* $\bar{A}$ *and* $\bar{B}$ *are right ideals of* A *and* B *respectively*, $f_{a+\bar{a},b+\bar{b}} = {}_{\bar{A}}f_{a,b}$; $\bar{f}_{a+\bar{a},b+\bar{b}} = {}_{\bar{B}}\bar{f}_{a,b}$ $\forall a \in A$, $\forall \bar{a} \in \bar{A}$, $\forall b \in B$, $\forall \bar{b} \in \bar{B}$.

Proof. Analogous to Proposition 4.

Def. 3. Let $N = A \oplus B$ be a direct sum of additive groups A and B; we define in N $\langle a, b \rangle \cdot \langle a', b' \rangle = \langle \lambda_{b'}(a), f_b(b') \rangle$, where $\lambda_b = \lambda(b)$ with $\lambda : B^+ \to F(A)^+$ is a homomorphism such that $\lambda_b \cdot \lambda_{b'} = \lambda_{f_{b'}(b)}$(1) and $f_b = \psi(b)$ with $\psi : B \setminus \{0\} \to \mathrm{AUT}(B^+)$ is a function such that $f_b \cdot f_{b'} = f_{f_b(b')}$(2) and $f_0 = \underline{0}$(3).

Proposition 8. *The structure* $N = [A \oplus B, \cdot]$, *as in* Def. 3, *is a left zero-symmetric near-ring in which* $A^0 = A_d(N)$ *and* ${}^0B$ *is an integral right ideal of* N; ${}^0B$ *is a two-sided ideal iff* $B = \ker \lambda$.

Proof. Easy verification.

The structure above mentioned will be called γ-*sum of* A *and* B and will be indicated by $A +_{\gamma} B$.

Theorem 3. *A near-ring* N *is a zero-symmetric near-ring with* $N^+ = A \oplus B$, *where* $A = A_d(N)$ *and* B *is a right ideal without zero divisors iff it is isomorphic to a* γ-*sum of* A *and* B.

Proof. Let N be a zero-symmetric near-ring with $N^+ = A \oplus B$, where $A = A_d(N)$ and B is a right ideal without zero divisors. Consider the homomorphism $\lambda : B^+ \to F(A)^+$ defined by: $\lambda(b) = \lambda_b : A \to A$ with $\lambda_b(a) = a \cdot b$ and the function $\psi : B \setminus \{0\} \to \mathrm{AUT}(B^+)$ defined by: $\psi(b) = \gamma_b$ and $\psi(0) = \underline{0}$. ψ fulfills (2) and (3) of Def. 3, moreover $(\lambda_b \cdot \lambda_{b'})(a) = \lambda_b(ab') = (ab')\, b = a(bb') = a\gamma_{b'}(b) = \lambda_{\gamma_{b'}(b)}(a)$ and (1) of Def. 3 holds.

Now we can easily prove that the correspondence $h : a + b \to \langle a, b \rangle$ from N to $A +_{\gamma} B$ is an isomorphism.

The converse follows directely from Proposition 8.

The multiplication of N infers a multiplication in A and B if we define $a \cdot a' = \pi_A(\langle a, 0\rangle \cdot \langle a', 0\rangle)$ and $b \cdot b' = \pi_B(\langle 0, b\rangle \cdot \langle 0, b'\rangle)$ and with respect to such operations A and B are near-rings.

Proposition 9. *Let A be a zero-near-ring, B an integral near-ring and $N = A +_\gamma B$. The multiplications inferred in A and B by multiplication of N and the multiplications of A and B coincide iff we define $f_b = \gamma_b$ $\forall b \in B$.*

Proof. Easy verification.

Proposition 10. *Let $N = A +_\gamma B$ be a zero-symmetric near-ring; a subset $\bar{N} = \bar{A} \times \bar{B}$ of N is:*

(a) *a subnear-ring of N iff $\bar{A}$ and $\bar{B}$ are subnear-rings of A and B respectively, $\lambda_{\bar{b}}(\bar{A}) \subseteq \bar{A}$ $\forall \bar{b} \in \bar{B}$, $f_b \in \mathscr{F}(\bar{B})$ $\forall b \in \bar{B}$;*

(b) *a left ideal of N iff $\bar{A}$ is a normal subgroup of A^+, $\bar{B}$ is a left ideal of B and $\lambda_{\bar{b}}(A) \subseteq \bar{A}$ $\forall \bar{b} \in \bar{B}$;*

(c) *a right ideal of N iff $\bar{A}$ is a normal subgroup of A^+, $\bar{B}$ is a right ideal of B and $\lambda_b(a + \bar{a}) =_{\bar{A}} \lambda_b(a)$ $\forall b \in B$, $\forall a \in A$, $\forall \bar{a} \in \bar{A}$.*

Proof. Analogous to Proposition 4.

References

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Summary

See Introduction.

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