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On Ikenberry's theorem (**)

1 - Introduction

Let $F(t, \boldsymbol{x}, \xi)$ be a molecular density at the place $\boldsymbol{x}$ and time t with velocity ξ, ξ and ξ_* be velocities of two particles before collision and ξ' and ξ'_* be their velocities after collision. Set $\boldsymbol{w} = \xi_* - \xi$, $\boldsymbol{w}' = \xi'_* - \xi'$, then $\boldsymbol{w}$ and $\boldsymbol{w}'$ are relative velocities before collision and after collision, respectiveli, with angle ϕ between them. Also let ε be the angle between the plane of $\boldsymbol{w}$ and $\boldsymbol{w}'$ and the plane containing $\boldsymbol{w}$ and a direction fixed in space, and let $S(\theta, w)$ be the scattering factor with $\theta = \frac{1}{2}(\pi - \phi)$. Now we can write the following expression for the Maxwell collisions operator [1]

(1) $$CF = \int (F'F'_* - FF_*)\, S(\theta, w) \sin\theta \, d\theta \, d\varepsilon \, d\xi_* .$$

In this formula F_*, F' and F'_* stand for F with its argument ξ replaced by ξ_*, ξ' and ξ'_*, respectively. Integration with respect to ξ_* is over three-dimensional velocity space and integrations with respect to θ and ε are from 0 to $\pi/2$ and 0 to 2π, respectively. For any function $g(t, \boldsymbol{x}, \xi)$, the total collisions operator is defined by

(2) $$\bar{C}g = \int g CF \, d\xi .$$

Using the properties of the total collisions operator, under certain conditions

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we can rewrite the operator (2) in the following form [1]

$$\bar{C}g = \tfrac{1}{2}\int FF_* \mathrm{B}g \, d\xi \, d\xi_* \tag{3}$$

$\mathrm{B}g$ being given by

$$\mathrm{B}g = \int (g' + g'_* - g - g_*) \, \mathrm{S}(\theta, w) \sin\theta \, d\theta \, d\varepsilon . \tag{4}$$

In the kinetic theory of gases, the problem of evaluating $\bar{C}g$ for certain function g is very important, because when we study the system of equation for moments or carry on approximate calculation for expansion of molecular density F, we need to evaluate the total collisions operator. In a gas of Maxwellian molecules, making use of the general structure of the total collision integrals, existence, uniqueness and a trend to equilibrium to the initial-value problem for Boltzmann equation can be given (cf. [1]).

The number density n and the velocity field $\boldsymbol{u}$ are defined as follows

$$n = \int F \, d\xi \qquad \boldsymbol{u} = \frac{1}{n}\int F\xi \, d\xi .$$

Let $\boldsymbol{c} = \xi - \boldsymbol{u}$, Ikenberry [2] introduced polynomials $\boldsymbol{Y}_s$ with components $Y_s = Y_{k_1 k_2 \dots k_s}$. They are harmonic functions obtained by subtracting from $c_{k_1} c_{k_2} \dots c_{k_s}$ that homogeneous symmetric polynomial of degree s in the components of $\boldsymbol{c}$ such as to annul the result of contracting the components of Y_s on any pair of indices. The first few Y_s are

$$Y(\boldsymbol{c}) = 1 \qquad Y_k(\boldsymbol{c}) = c_k \qquad Y_{km}(\boldsymbol{c}) = c_k c_m - \frac{c^2}{3}\delta_{km}$$

$$Y_{kmr}(\boldsymbol{c}) = c_k c_m c_r - \frac{3}{5} c^2 c_{(k}\delta_{mr)} \qquad Y_{kmrs}(\boldsymbol{c}) = c_k c_m c_r c_s - \frac{6}{7} c^2 c_{(k} c_m \delta_{rs)} + \frac{4}{35} c^4 \delta_{(km}\delta_{rs)}$$

in which, parentheses around a set of s subscripts indicates the sum over the $s!$ permutations of the indices, divided by $s!$. If $A_{k_1 \dots k_s}$ is an s^{th}-order tensor, $A_{(k_1 \dots k_s)}$ is the totally symmetric tensor obtained by symmetrizing $A_{k_1 \dots k_s}$. $A_{\{k_1 \dots k_s\}}$ denotes the totally symmetric traceless tensor constructed form $A_{k_1 \dots k_s}$. It is not difficulty to derive the following general formula for $A_{\{k_1 \dots k_s\}}$

$$A_{\{k_1 \dots k_s\}} = \sum_{q=0}^{[\frac{1}{2}s]} b_q^s A^{(q)}_{(k_1 \dots k_{s-2q}} \delta_{k_{s-2q+1} k_{s-2q+2}} \dots \delta_{k_{s-1} k_s)} \tag{5}$$

where

$$b_q^s = (-1)^q \frac{s!\,(2s-2q+1)!!\,(2s+1)}{(s-2q)!\,(2q)!!\,(2s+1)!!\,(2s-2q+1)} \tag{6}$$

$$A^{(q)}_{k_1 \ldots k_{s-2q}} = A_{(k_1 \ldots k_{s-2q} m_1 m_1 \ldots m_q m_q)}\,.$$

Using these, we obtain [1]

$$Y_s(\boldsymbol{c}) = Y_{k_1 \ldots k_s}(\boldsymbol{c}) = c_{\{k_1} \ldots c_{k_s\}} = \sum_{q=0}^{[\frac{1}{2}s]} b_q^s c^{2q} c_{(k_1} \ldots c_{k_{s-2q}} \delta_{k_{s-2q+1} k_{s-2q+2}} \ldots \delta_{k_{s-1} k_s)}\,. \tag{7}$$

In [2], homogeneous polynomials Y_{2r+s} of degree $2r+s$ are defined by

$$\boldsymbol{Y}_{2r|s}(\boldsymbol{c}) = c^{2r} \boldsymbol{Y}_s(\boldsymbol{c})\,. \tag{8}$$

These polynomials form a complete set: any symmetric polynomial can be expressed uniquely as a linear combination of them. For example [1]

$$c_{k_1} \ldots c_{k_s} = \sum_{q=0}^{[\frac{1}{2}s]} a_q^s Y_{2q|(k_1 \ldots k_{s-2q}}(\boldsymbol{c})\, \delta_{k_{s-2q+1} k_{s-2q+2}} \ldots \delta_{k_{s-1} k_s)} \tag{9}$$

a_q^s being given by

$$a_q^s = \frac{s!\,(2s-4q+1)!!}{(s-2q)!\,(2q)!!\,(2s-2q+1)!!}\,. \tag{10}$$

Corresponding to each polynomial $Y_{2r|s}$, the spherical moment $\mathsf{P}_{2r|s}$ is defined as follows

$$\mathsf{P}_{2r|s} = \mathsf{m} \int F \boldsymbol{Y}_{2r|s}(\boldsymbol{c})\, d\xi$$

m being molecular mass. The sum $2r+s$ is the order of the spherical moment $\mathsf{P}_{2r|s}$.

Ikenberry [2] revealed the structure of collision integrals for a gas of Maxwellian molecules. He proved that $\mathfrak{C}\boldsymbol{Y}_{2r|s} = -\,\mathsf{c}_{2r|s} \mathsf{P}_{2r|s}$ plus a bilinear combination of the spherical moments of lower orders, the sum of the orders in each term being $2r+s$. He evaluated explicitly only the coefficient $\mathsf{c}_{2r|s}$ and did not obtain the coefficients in the bilinear combination. In this paper, the coefficients in the bilinear combination are given explicitly. We shall see that the expressions of these coefficients are very complex. First we state

Ikenberry's Theorem. *In a gas of Maxwellian molecules, if F possesses moments of order* $1, \ldots, 2r+s$ *and is as to render* (3) *valid when* $g = Y_{2r|s}$, *then*

$$\mathsf{m}\bar{\mathsf{c}}\boldsymbol{Y}_{2r|s} = -\mathsf{c}_{2r|s}\mathsf{P}_{2r|s} + \mathsf{Q}_{2r|s} \qquad 2r+s \geqslant 1\,.$$

$\mathsf{Q}_{2r|s}$ *is a bilinear function spherical moments the orders of which are positive numbers whose sum is* $2r+s$

$$\mathsf{Q}_{2r|s} = \sum \mathsf{C}_{r_1r_2|s_1s_2s}\,\mathsf{P}_{2r_1|s_1}\,\mathsf{P}_{2r_2|s_2}$$

$$2r_1+s_1 \geqslant 2r_2+s_2 > 0 \qquad 2r_1+s_1+2r_2+s_2 = 2r+s\,.$$

the tensorial coefficient $\mathsf{C}_{r_1r_2|s_1s_2s}$ *is a function of* m *and* g *alone, and the scalar coefficients* $\mathsf{c}_{2r|s}$ *is given as follows*

$$\mathsf{c}_{2r|s} = 2\pi\int_0^{\pi/2}(1-\cos^{2r+s}\theta\mathsf{P}_s(\cos\theta) - \sin^{2r+s}\theta\mathsf{P}_s(\sin\theta))\,\mathsf{S}(\theta)\sin\theta\,d\theta$$

in which P_s *denotes the Legendre polynomial of order* s.

In the next section, we give some formulae which will be used in the calculation of $\mathsf{Q}_{2r|s}$. In the last section, we obtain a refined form of Ikenberry's theorem which include explicit expression $\mathsf{Q}_{2r|s}$.

2 - Basic formulae

Set $\boldsymbol{v} = \boldsymbol{c}_* + \boldsymbol{c}$, $\boldsymbol{w} = \boldsymbol{c}_* - \boldsymbol{c}$, from the laws of momentum and energy it follows that $\boldsymbol{v} = \boldsymbol{v}'$, $\boldsymbol{w} = \boldsymbol{w}'$. In order to evaluate $\mathsf{Q}_{2r|s}$, we need the following formulae:

Formula 1.

$$\int \boldsymbol{Y}_s(\boldsymbol{w}')\,d\varepsilon = 2\pi\mathsf{P}_s(\cos\phi)\,\boldsymbol{Y}_s(\boldsymbol{w})\,. \tag{14}$$

Formula 2.

$$c'^{2r}c'_{k_1}\ldots c'_{k_s} + c'^{2r}_{*}c'_{*k_1}\ldots c'_{*k_s} = \sum_{p=0}^{r}\sum_{q=0}^{s} d^{r,s}_{p,q}(v^2+w^2)(\boldsymbol{v}\cdot\boldsymbol{w}')\,w'_{(k_1}\ldots w'_{k_q}v_{k_{q+1}}\ldots v_{k_s)} \tag{15}$$

$d_{p,q}^{r,s}$ being given by

$$d_{p,q}^{r,s} = \frac{1}{2^{2r+s-p}} \binom{r}{p} \binom{s}{q} (1 + (-1)^{p+q}). \tag{16}$$

Obviously, when $p+q$ is odd, $d_{p,q}^{r,s} = 0$.

Formula 3. Let p_1, p_2 and p_3 be non-negative integers. If $r - p_1$, $p_1 - p_2$ and $p_2 - p_3$ are non-negative, then

$$(v^2 + w^2)^{r-p_1} (\boldsymbol{v} \cdot \boldsymbol{w})^{p_1-p_2} v^{2p_3} w^{2(p_2-p_3)} = \sum_{p=0}^{p_2} \sum_{q=0}^{r-p} e_{p,q}^{r,p_1,p_2,p_3} c^{2q} c_*^{2(r-p-q)} (\boldsymbol{c} \cdot \boldsymbol{c}_*)^p \tag{17}$$

in which

$$e_{p,q}^{r,p_1,p_2,p_3} \tag{18}$$

$$= \sum_{i=\max(0,p-p_3)}^{\min(p_2-p_3,p)} \sum_{j=\max(0,p+q+p_1-r-p_2)}^{\min(p_1-p_2,p)} (-1)^{i+j} 2^{r+p-p_1} \binom{p_3}{p-i} \binom{p_2-p_3}{i} \binom{p_1-p_2}{j} \binom{r+p_2-p_1-p}{q-j}.$$

Formula 4.

$$w_{(k_1} \cdots w_{k_q} v_{k_{q+1}} \cdots v_{k_s)} = \sum_{p=0}^{s} f_p^{s,q} c_{(k_1} \cdots c_{k_p} c_{*k_{p+1}} \cdots c_{*k_s)} \quad \text{where} \tag{19}$$

$$f_p^{s,q} = \sum_{i=\max(0,p+q-s)}^{\min(q,p)} (-1)^i \binom{q}{i} \binom{s-q}{p-i}. \tag{20}$$

Formula 5. Let $A_{n_1 \ldots n_{p+q-2q_1}}$ and $B_{m_1 \ldots m_p}$ be two symmetric tensors, whose orders are $p+q-2q_1$ and p respectively. Set $l_i = k_i$ $(1 \leqslant i \leqslant q)$, $l_{i+q} = m_i$ $(1 \leqslant i \leqslant p)$. If $p \geqslant q_1$, then

$$B_{m_1 \ldots m_p} A_{(l_1 \ldots l_{p+q-2q_1}} \delta_{l_{p+q-2q_1+1} l_{p+q-2q_1+2}} \cdots \delta_{l_{p+q-1} l_{p+q})} \tag{21}$$

$$= \sum_{p_1=\max(q_1, 2q_1-q)}^{\min(2q_1,p)} g_{p_1}^{p,q,q_1} A_{m_1 \ldots m_{p-p_1}(k_1 \ldots k_{q+p_1-2q_1}} B_{k_{q+p_1-2q_1+1} \ldots k_q) m_1 \ldots m_{p-p_1} a_1 a_1 \ldots a_{p_1-q_1} a_{p_1-q_1}} + \mathrm{R}$$

in which

$$g_{p_1}^{p,q,q_1} = \frac{1}{\binom{p+q}{2q_1} (2q_1-1)!!} \binom{p}{p_1} \binom{q}{2q_1-p_1} \binom{p_1}{2q_1-p_1} (2q_1-p_1)! \, (2p_1-2q_1-1)!!. \tag{22}$$

In the expression (21), R denotes a sum each term of which includes at least one $\delta_{l_i l_j}$ $(1 \leqslant i,\ j \leqslant q)$, that is to say δ has tensorial indices $k_i k_j$.

We often use the following three special cases:

Case 1. *If* $B_{m_1 \dots m_p} = b_{m_1} b_{m_2} \dots b_{m_p}$, *then*

$$b_{m_1} \dots b_{m_p} A_{(l_1 \dots l_{p+q-2q_1}} \delta_{l_{p+q-2q_1+1} l_{p+q-2q_1+2}} \dots \delta_{l_{p+q-1} l_{p+q})} \tag{23}$$

$$= \sum_{p_1=\max(q_1, 2q_1-q)}^{\min(2q_1, p)} g_{p_1}^{p,q,q_1} b^{2(p_1-q_1)} b_{m_1} \dots b_{m_{p-p_1}} A_{m_1 \dots m_{p-p_1}(k_1 \dots k_{q+p_1-2q_1}} b_{k_{q+p_1-2q_1+1}} \dots b_{k_q)} + \mathsf{R}\,.$$

Case 2. *If* $B_{m_1 \dots m_p} = b_{m_1} \dots b_{m_p}$, $A_{n_1 \dots n_{p+q-2q_1}} = a_{n_1} \dots a_{p+q-2q_1}$, *then*

$$b_{m_1} \dots b_{m_p} a_{(l_1} \dots a_{l_{p+q-2q_1}} \delta_{l p+q-2q_1+1} l_{p+g-2q_1+2} \dots \delta_{l_{p+q-1} l_{p+g})}$$

$$= \sum_{p_1=\max(q_1, 2q_1-q)}^{\min(2q_1, p)} g_{p_1}^{p,q,q_1} b^{2(p_1-q_1)} (\boldsymbol{b} \cdot \boldsymbol{a})^{p-p_1} a_{(k_1} \dots a_{k_{q+p_1-2q+1}} b_{k_{q+p_1-2q_1+1}} \dots b_{k_q)}) + \mathsf{R}\,.$$

Case 3. *If* $B_{m_1 \dots m_p}$ *is a traceless tensor, then*

$$B_{m_1 \dots m_p} A_{(l_1 \dots l_{p+q-2q_1}} \delta_{l_{p+q-2q_1+1} l_{p+q-2q_1+2}} \dots \delta_{l_{p+q-1} l_{p+g})} \tag{25}$$

$$= \delta(q-q_1)\, g_{q_1}^{p,q,q_1} A_{m_1 \dots m_{p-q_1}(k_1 \dots k_{q-q_1}} B_{k_{q-q_1+1} \dots k_q) m_1 \dots m_{p-q_1}} + \mathsf{R}$$

the function $\delta(x)$ being defined by $\delta(x) = \begin{cases} 1 & x \geqslant 0 \\ 0 & x < 0. \end{cases}$

Proof. First, Formula 1 can be found in paper [2].
Using $\boldsymbol{v} = \boldsymbol{c}_* + \boldsymbol{c}$, $\boldsymbol{w} = \boldsymbol{c}_* - \boldsymbol{c}$, $\boldsymbol{v} = \boldsymbol{v}'$ and $\boldsymbol{w} = \boldsymbol{w}'$, we find that

$$c'^{2r} c'_{k_1} \dots c'_{k_s} + c'^{2r}_* c'_{*k_1} \dots c'_{*k_s} = \frac{1}{2^{2r+s}} [(v^2 + w^2 - 2\boldsymbol{v} \cdot \boldsymbol{w}')^r (v_{k_1} - w'_{k_1}) \dots (v_{k_s} - w'_{k_s})$$

$$+ (v^2 + w^2 + 2\boldsymbol{v} \cdot \boldsymbol{w}')^r (v_{k_1} + w'_{k_1}) \dots (v_{k_s} + w'_{k_s})]\,.$$

Next, we expand the terms $(v^2 + w^2 - 2\boldsymbol{v} \cdot \boldsymbol{w}')^r$ and $(v^2 + w^2 + 2\boldsymbol{v} \cdot \boldsymbol{w}')^r$ by using the binomial theorem

$$(v^2+w^2-2\boldsymbol{v}\cdot\boldsymbol{w}')^r=\sum_{p=0}^{r}\binom{r}{p}(-1)^p(v^2+w^2)^{r-p}(2\boldsymbol{v}\cdot\boldsymbol{w}')^p$$

$$(v^2+w^2+2\boldsymbol{v}\cdot\boldsymbol{w}')^r=\sum_{p=0}^{r}\binom{r}{p}(v^2+w^2)^{r-p}(2\boldsymbol{v}\cdot\boldsymbol{w}')^p\,.$$

For the expressions $(v_{k_1}-w'_{k_1})\dots(v_{k_s}-w'_{k_s})$ and $(v_{k_1}+w'_{k_1})\dots(v_{k_s}+w'_{k_s})$, it is not difficult to see that following variant of the binomial theorem is valid:

$$(v_{k_1}-w'_{k_1})\dots(v_{k_s}-w'_{k_s})=\sum_{q=0}^{s}\binom{s}{q}(-1)^q w_{(k_1}\dots w_{k_q}v_{k_{q+1}}\dots v_{k_s)}$$

$$(v_{k_1}+w'_{k_1})\dots(v_{k_s}+w'_{k_s})=\sum_{q=0}^{s}\binom{s}{q}w_{(k_1}\dots w_{k_q}v_{k_{q+1}}\dots v_{k_s)}\,.$$

Combining these expressions, we easily obtain Formula 2.

For Formula 3, we have

$$(v^2+w^2)^{r-p_1}(\boldsymbol{v}\cdot\boldsymbol{w})^{p_1-p_2}v^{2p_3}w^{2(p_2-p_3)}$$

$$=[2(c^2+c_*^2)]^{r-p_1}(c_*^2-c^2)^{p_1-p_2}(c_*^2+c^2+2\boldsymbol{c}\cdot\boldsymbol{c}_*)^{p_3}(c_*^2+c^2-2\boldsymbol{c}\cdot\boldsymbol{c}_*)^{p_2-p_3}$$

$$=\sum_{i=0}^{p_2-p_3}\sum_{k=0}^{p_3}\binom{p_2-p_3}{i}\binom{p_3}{k}(-1)^i 2^{r-p_1+i+k}(c_*^2-c^2)^{p_1-p_2}(c_*^2+c^2)^{r+p_2-p_1-i-k}(\boldsymbol{c}\cdot\boldsymbol{c}_*)^{i+k}$$

$$=\sum_{i=0}^{p_2-p_3}\sum_{k=0}^{p_3}\sum_{j=0}^{p_1-p_2}\sum_{l=0}^{r+p_2-p_1-i-k}\binom{p_2-p_3}{i}\binom{p_3}{k}\binom{p_1-p_2}{j}\binom{r+p_2-p_1-i-k}{l}$$

$$\cdot(-1)^{i+j}2^{r-p_1+i+k}c^{2(j+l)}c_*^{2(r-i-k-j-l)}(\boldsymbol{c}\cdot\boldsymbol{c}_*)^{i+k}\,.$$

In the above expression, set $p=i+k$, $q=j+k$ to obtain

(26) $$(v^2+w^2)^{r-p_1}(\boldsymbol{v}\cdot\boldsymbol{w})^{p_1-p_2}v^{2p_3}w^{2(p_2-p_3)}$$

$$=\sum_{i=0}^{p_2-p_3}\sum_{p=0}^{p_3+i}\sum_{j=0}^{p_1-p_2}\sum_{q=j}^{r+p_2-p_1-p+j}\binom{p_3}{p-i}\binom{p_2-p_3}{i}\binom{p_1-p_2}{j}\binom{r+p_2-p_1-p}{q-j}$$

$$\cdot(-1)^{i+j}2^{r+p-p_1}c^{2q}c_*^{2(r-p-q)}(\boldsymbol{c}\cdot\boldsymbol{c}_*)^p\,.$$

Next, we exchange the orders of the following summations

$$\sum_{i=0}^{p_2-p_3}\sum_{p=i}^{p_3+i}=\sum_{p=0}^{p_2}\sum_{i=\max(0,p-p_3)}^{\min(p_2-p_3,p)}\qquad \sum_{j=0}^{p_1-p_2}\sum_{q=j}^{r+p_2-p_1-p+j}=\sum_{q=0}^{r-p}\sum_{j=\max(0,p+q+p_1-r-p_2)}^{\min(p_1-p_2,q)}.$$

Putting these two expressions into (26) and rearranging, we obtain the Formula 3.

Using $\boldsymbol{w}=\boldsymbol{c}_*-\boldsymbol{c}$ and $\boldsymbol{v}=\boldsymbol{c}_*+\boldsymbol{c}$, and multiplying out the symmetric tensor $w_{(k_1}\dots w_{k_q}v_{k_{q+1}}\dots v_{k_s)}$ we can show that the equality (19) must hold. Now, it remains to calculate the coefficients of $c_{(k_1}\dots c_{k_p}c_{*k_{p+1}}\dots c_{*k_s)}$. We take i components of $\boldsymbol{c}$ out of the components of $\boldsymbol{w}$ and $p-i$ components of $\boldsymbol{c}$ out of the components of $\boldsymbol{v}$, the total of which is $\binom{q}{i}\binom{s-q}{p-i}$ with $0\leqslant i\leqslant q$ and $0\leqslant p-i\leqslant s-q$, of which the sign is $(-1)^i$. Hence, (20) is valid.

Finally, we prove Formula 5. Since $A_{n_1\dots n_{p+q-2q_1}}$ is a symmetric tensor, $\delta_{(n_1n_2}\dots\delta_{n_{2q_1-1}n_{2q_1})}$ has $\dfrac{(2q_1)!!}{2^{q_1}q_1!}=(2q_1-1)!!$ different terms; hence $A_{(l_1\dots l_{p+q-2q_1}}$ $\delta_{l_{p+q-2q_1+1}l_{p+q-2q_1+2}}\dots\delta_{l_{p+q-1}l_{p+q})}$ has $\binom{p+q}{2q_1}(2q_1-1)!!$ different terms, these terms can be divided into two classes, one including those terms in which two indices of each δ have at least one out of the values $m_1,\ m_2,\dots m_p$, the other including those terms each of which has at least one $\delta_{l_il_j}$ $(1\leqslant i,\ j\leqslant q)$. When terms in the second class muliplied by $B_{m_1\dots m_p}$ are summed, they obviuosly become the expression R in the right hand of (21), and when terms in the first class multiplied by $B_{m_1\dots m_p}$ are summed, the result does not include δ. Now we calculate the total number of the terms of the first class. First, we take $2q_1$ indices out of $l_1,\dots,l_{p+q}$, in which p_1 indices are taken out of $m_1,\dots,m_p$, and $2q_1-p_1$ indices are taken out of $k_1,\dots,k_q$. Obviously, they must satisfy that $0\leqslant p_1\leqslant p$, and $0\leqslant 2q_1-p_1\leqslant q$. The total way is $\binom{p}{p_1}\binom{q}{2q_1-p_1}$. We put these $2q_1$ indices on $q_1\,\delta$, and other $p+q-2q_1$ indices on tensor $A_{n_1\dots n_{p+q-2q_1}}$. In order to obtain the first class, we meat have $q_1\leqslant p_1$ and each δ at most has one index out of $k_1,\dots,k_q$. From the three inequalities mentioned above, we obtain that $\max(q_1,2q_1-q)\leqslant p_1\leqslant\min(p,2q_1)$. By using the principle of multiplication, the total number of the terms of the first class for fixed p_1 are $\binom{p}{p_1}\binom{q}{2q_1-p_1}\binom{p_1}{2q_1-p_1}(2q_1-p_1)!\,(2p_1-2q_1-1)!!$.

Finally, summing the products of $B_{m_1\dots m_p}$ by the terms of the first class and using the definition of parentheses around a set of indices, we readily derive the following formula

$$B_{m_1\dots m_p}A_{(l_1\dots l_{p+q-2q_1}}\delta_{l_{p+q-2q_1+1}\,l_{p+q-2q_1+2}}\cdots\delta_{l_{p+q-1}\,l_{p+q})}$$

$$=\sum_{p=\max(q_1,2q_1-q)}^{\min(2q_1,p)} g_{p_1}^{p,q,q_1}A_{m_1\dots m_{p-p_1}(k_1\dots k_{q+p_1-2q_1}}B_{k_{q+p_1-2q_1+1}\dots k_q)m_1\dots m_{p-p_1}a_1a_1\dots a_{p_1-q_1}a_{p_1-q_1}}+\mathrm{R}$$

in which

$$g_{p_1}^{p,q,q_1}=\frac{1}{\binom{p+q}{2q_1}(2q_1-1)!!}\binom{p}{p_1}\binom{q}{2q_1-p_1}\binom{p_1}{2q_1-p_1}(2q_1-p_1)!\,(2p_1-2q_1-1)!!\,.$$

This is precisely the expressions (21) and (22). Hence we have completed the proof of the Formula 1 to Formula 5.

We have the following properties for the parentheses and the braces around a set of indices:

Property 1.

(27) $$A_{\{k_1\dots k_{s-2}}\delta_{k_{s-1}k_s\}}=0\,.$$

Property 2. *If $A_{k_1\dots k_s}$ is an s^{th}-order totally symmetric traceless tensor, then*

(28) $$A_{\{k_1\dots k_s\}}=A_{k_1\dots k_s}\,.$$

Property 3. *If $C_{k_1\dots k_s}=A_{k_1\dots k_s}+B_{k_1\dots k_s}$, then*

(29) $$C_{(k_1\dots k_s)}=A_{(k_1\dots k_s)}+B_{(k_1\dots k_s)}$$

(30) $$C_{\{k_1\dots k_s\}}=A_{\{k_1\dots k_s\}}+B_{\{k_1\dots k_s\}}\,.$$

Property 4.

(31) $$A_{((k_1\dots k_q)k_{q+1}\dots k_s)}=A_{(k_1\dots k_s)}\,.$$

From (5) and (6) it is not difficult to derive (27), and (28)-(31) are obvious.

3 - Calculation of $Q_{2r|s}$

Having obtained the basic formulae stated in the proceeding section, we can proceed to calculate $Q_{2r|s}$. From (3) and (4), we know that the unique difficulty in calculating $Q_{2r|s}$ is to evaluate the integral $\int (Y_{2r|s}(c') + Y_{2r|s}(c'_*))\, d\varepsilon$. We see from the proof of Theorem 1 in Chapter XIV of [1] that if g is a polynomial of degree m in the components of $\boldsymbol{c}$, then $\int (g(\boldsymbol{c}') + g(\boldsymbol{c}_*))\, d\varepsilon$ is a homogeneous polynomial of degree m in the components of $\boldsymbol{c}$ and $\boldsymbol{c}_*$. Thus if $g = Y_{2r|s}(\boldsymbol{c})$, $\int (Y_{2r|s}(\boldsymbol{c}') + Y_{2r|s}(\boldsymbol{c}'_*))\, d\varepsilon$ is a homogeneous polynomial of degree $2r+s$, which is also an s^{th}-order symmetric tensor, and it must have the following form

$$\int_0^{2\pi} (Y_{2r|s}(\boldsymbol{c}') + Y_{2r|s}(\boldsymbol{c}'_*))\, d\varepsilon = \sum B^{r,s}_{l,n,k,p}\, c^{2l} c_*^{2n} (\boldsymbol{c}\cdot\boldsymbol{c}_*)^k c_{(k_1} \dots c_{k_p} c_{*k_{p+1}} \dots c_{*k_s)} + \mathsf{R}$$

where $l+n+k=r$, $B^{r,s}_{l,n,k,p}$ is a constant and R is a homogeneous polynomial of degree $2r+s$ each of which is an s^{th}-order tensor, which includes at least one $\delta_{k_ik_j}$ $(1 \leqslant i,\ j \leqslant s)$. In the rest of the paper, R always has this meaning, although in each instance R is different. Since the left-hand side of the above expression is traceless, using (28), (30) and (27) we find that

$$\int_0^{2\pi} (Y_{2r|s}(\boldsymbol{c}') + Y_{2r|s}(\boldsymbol{c}'_*))\, d\varepsilon = \sum B^{r,s}_{l,n,k,p}\, c^{2l} c_*^{2n} (\boldsymbol{c}\cdot\boldsymbol{c}_*)^k c_{\{k_1} \dots c_{k_p} c_{*k_{p+1}} \dots c_{*k_s\}}\,.$$

We see from that we need only calculate the terms which do not include $\delta_{k_ik_j}$ $(1 \leqslant i,\ j \leqslant s)$. In virtue of (15), we have

$$(32)\qquad \int_0^{2\pi} (Y_{2r|s}(\boldsymbol{c}') + Y_{2r|s}(\boldsymbol{c}'_*))\, d\varepsilon$$

$$= \int_0^{2\pi} (c'^{2r} c'_{k_1} \dots c'_{k_s} + c'^{2r}_* c'_{*k_1} \dots c'_{*k_s})\, d\varepsilon + \mathsf{R}$$

$$= \sum_{p_1=0}^{r} \sum_{q_1=0}^{s} d^{r,s}_{p_1,q_1} (v^2+w^2)^{r-p_1} \int_0^{2\pi} (\boldsymbol{v}\cdot\boldsymbol{w}')^{p_1} w'_{(k_1} \dots w'_{k_q} v_{k_{q+1}} \dots v_{k_s)}\, d\varepsilon + \mathsf{R}$$

$d^{r,s}_{p,q}$ being given by (16).

Now we proceed to evaluate the integrals in the right-hand side of (32). Set $l_i = k_i$ $(1 \leqslant i \leqslant q_1)$, $l_{i+q_1} = m_i$ $(1 \leqslant i \leqslant p_1)$; in view of (7), (9) and (14), we find that

$$\int_0^{2\pi} (\boldsymbol{v}\cdot\boldsymbol{w}')^{p_1} w'_{k_1} \dots w'_{k_{q_1}}\, d\varepsilon = \int_0^{2\pi} v_{m_1} \dots v_{m_{p_1}} w'_{k_1} \dots w'_{k_{q_1}} w'_{m_1} \dots w'_{m_{p_1}}\, d\varepsilon\,.$$

$$= v_{m_1} \dots v_{m_{p_1}} \int_0^{2\pi} \sum_{p_2=0}^{[\frac{1}{2}(p_1+q_1)]} a_{p_2}^{p_1+q_1} w^{2p_2} Y_{(l_1 \dots l_{p_1+q_1-2p_2}}(\boldsymbol{w}') \delta_{l_{p_1+q_1-2p_2+1}\, l_{p_1+q_1-2p_2+2}} \dots \delta_{l_{p_1+q_1-1}\, l_{p_1+q_1})}$$

$$= 2\pi \sum_{p_2=0}^{\min([\frac{1}{2}(p_1+q_1)], p_1)} \mathsf{P}_{p_1+q_1-2p_2}(\cos\phi)\, a_{p_2}^{p_1+q_1} w^{2p_2} v_{m_1} \dots v_{m_{p_1}}$$

$$\times Y_{(l_1 \dots l_{p_1+q_1-2p_2}}(\boldsymbol{w}) \delta_{l_{p_1+q_1-2p_2+1}\, l_{p_1+q_1-2p_2+2}} \dots \delta_{l_{p_1+q_1-1}\, l_{p_1+q_1})} + \mathsf{R}$$

$$= 2\pi \sum_{p_2=0}^{\min([\frac{1}{2}(p_1+q_1)], p_1)} \sum_{q_2=0}^{\min([\frac{1}{2}(p_1+q_1)]-p_2, p_1-p_2)} \mathsf{P}_{p_1+q_1-2p_2}(\cos\phi)\, a_{p_2}^{p_1+q_2} b_{q_2}^{p_1+q_1-2p_2} w^{2(p_2+q_2)}$$

$$\times v_{m_1} \dots v_{m_{p_1}} w_{(l_1} \dots w_{l_{p_1+q_1-2(p_2+q_2)}} \delta_{l_{p_1+q_1-2(p_2+q_2)+1}\, l_{p_1+q_1-2(p_2+q_2)+2}} \dots \delta_{l_{p_1+q_1-1}\, l_{p_1+q_1})} + \mathsf{R}.$$

a_q^s and b_q^s being given by (10) and (6), P_q being the Legendre polynomial of order q.

In the above expression, the upper limits of summation indices p_2 and q_2 have been changed from $[\frac{1}{2}(p_1+q_1)]$ and $[\frac{1}{2}(p_1+q_1)]-p_2$ to $\min([\frac{1}{2}(p_1+q_1)],\ p_1)$ and $\min([\frac{1}{2}(p_1+q_1)]-p_2,\ p_1-p_2)$, respectively because the terms with $p_2 > p_1$ and $q_2 > p_1 - p_2$ include at least one $k_i k_j$ $(1 \leq i, j \leq s)$ and so are included in expression R. Using (24), we obtain

$$\int_0^{2\pi} (\boldsymbol{v} \cdot \boldsymbol{w}')^{p_1} w'_{k_1} \dots w'_{k_{q_1}}\, \mathrm{d}\varepsilon \tag{33}$$

$$= 2\pi \sum_{p_2=0}^{\min([\frac{1}{2}(p_1+q_1)], p_1)} \sum_{q_2=0}^{\min([\frac{1}{2}(p_1+q_1)]-p_2, p_1-p_2)} \sum_{n=\max(p_2+q_2, 2(p_2+q_2)-q_1)}^{\min(2(p_2+q_2), p_1)} p_{p_1+q_1-2p_2}(\cos\phi)\, a_{p_2}^{p_1+q_1} b_{q_2}^{p_1+q_1-2p_2} g_n^{p_1,q_1,p_2+q_2}$$

$$\times w^{2(p_2+q_2)} v^{2(n-p_2-q_2)} (\boldsymbol{v} \cdot \boldsymbol{w})^{p_1-n} w_{(k_1} \dots w_{k_{q_1+n-2(p_2+q_2)}} v_{k_{q_1+n-2(p_2+q_2)+1}} \dots v_{k_{q_1})} + \mathsf{R}$$

$g_{p_1}^{p,q,q_1}$ being given by (22).

Putting (33) into (32) and using (29), (31), (17) and (19), we see that

$$\int_0^{2\pi} (Y_{2r|s}(\boldsymbol{c}') + Y_{2r|s}(\boldsymbol{c}'_*))\, \mathrm{d}\varepsilon \tag{34}$$

$$= \sum_{p_1=0}^{r} \sum_{q_1=0}^{s} \sum_{p_2=0}^{\min([\frac{1}{2}(p_1+q_1)], p_1)} \sum_{q_2=0}^{\min([\frac{1}{2}(p_1+q_1)]-p_2, p_1-p_2)} \sum_{n=\max(p_2+q_2, 2(p_2+q_2)-q_1)}^{\min(2(p_2+q_2), p_1)} 2\pi\, \mathsf{P}_{p_1+q_1-2p_2}(\cos\phi)\, a_{p_2}^{p_1+q_1}$$

$$\times b_{q_2}^{p_1+q_1-2p_2} d_{p_1,q_1}^{r,s} g_n^{p_1,q_1,p_2+q_2} (w^2+v^2)^{r-p_1} (\boldsymbol{v} \cdot \boldsymbol{w})^{p_1-n} w^{2(p_2+q_2)} v^{2(n-p_2-q_2)}$$

$$\times w_{(k_1} \dots w_{k_{q_1+n-2(p_2+q_2)}} v_{k_{q_1+n-2(p_2+q_2)+1}} \dots v_{k_s)} + \mathsf{R}$$

$$= \sum_{p_1=0}^{r} \sum_{q_1=0}^{s} \sum_{p_2=0}^{\min([\frac{1}{2}(p_1+q_1)],\, p_1)} \sum_{q_2=0}^{\min([\frac{1}{2}(p_1+q_1)]-p_2,\, p_1-p_2)} \sum_{n=\max(p_2+q_2,\, 2(p_2+q_2)-q_1)}^{\min(2(p_2+q_2),\, p_1)} \sum_{p_3=0}^{n} \sum_{q_3=0}^{r-p_3} \sum_{t=0}^{s} 2\pi \mathsf{P}_{p_1+q_1-2p_2}(\cos\phi)$$

$$\times\, a_{p_2}^{p_1+q_1}\, b_{q_2}^{p_1+q_1-2p_2}\, d_{p_1,q_1}^{r,s}\, e_{p_3,q_3}^{r,p_1,n,n-p_2-q_2}\, f_t^{s,q_1+n-2(p_2+q_2)}\, g_n^{p_1,q_1,p_2+q_2}$$

$$\times\, c^{2q_3}\, c_*^{2(r-p_2-p_3)}\, (\boldsymbol{c}\cdot\boldsymbol{c}_*)^{p_3}\, c_{(k_1}\ldots c_{k_t}\, c_{*k_{t+1}}\ldots c_{*k_s)} + \mathsf{R}$$

$e_{p,q}^{r,p,p,p}$ and $f_p^{s,q}$ being given by (18) and (20).

In order to simplify the expression (34), we shall use the following formulae, which are easily verified

(35) $$\sum_{n=\max(p_2+q_2,\, 2(p_2+q_2)-q_1)}^{\min(2(p_2+q_2),\, p_1)} \sum_{p_3=0}^{n} = \sum_{p_3=0}^{\min(2(p_2+q_2),\, p_1)} \sum_{n=\max(p_3,\, p_2+q_2,\, 2(p_2+q_2)-q_1)}^{\min(2(p_2+q_2),\, p_1)}$$

(36) $$\sum_{q_2=0}^{\min([\frac{1}{2})p_1+q_1)]-p_2,\, p_1-p_2)} \sum_{p_3=0}^{\min(2(p_2+q_2),\, p_1)} = \sum_{P_3=0}^{\min(2[\frac{1}{2}(p_1+q_1)],\, p_1)} \sum_{q_2=\max(0,\, [\frac{1}{2}(p_3+1)]-p_2)}^{\min([\frac{1}{2}(p_1+q_2)]-p_2,\, p_1-p_2)}$$

(37) $$\sum_{q_1=0}^{s} \sum_{p_3=0}^{\min(2[\frac{1}{2}(p_1+q_1)],\, p_1)} = \sum_{p_2=0}^{\min(2[\frac{1}{2}(p_1+s)],\, p_1)} \sum_{q_1=\max(0,\, p_3-2[\frac{p_1}{2}])}^{s}$$

(38) $$\sum_{p_1=0}^{r} \sum_{p_3=0}^{\min(2[\frac{1}{2}(p_1+s)],\, p_1)} = \sum_{p_3=0}^{\min(2[\frac{1}{2}(r+s)],\, r)} \sum_{p_1=\max(p_3,\, 2[\frac{1}{2}(p_3-s+1)])}^{r} .$$

Set

(39) $$B_{p_3,q_3,t}^{r,s} = \sum_{p_1=\max(p_3,\, 2[\frac{1}{2}p_3-s+1)])}^{r} \sum_{q_1=\max(0,\, p_3-2[\frac{p_1}{2}])}^{s} \sum_{p_2=0}^{\min([\frac{1}{2}(p_1+q_1)],\, p_1)} \sum_{q_2=\max(0,\, [\frac{1}{2}(p_3+1)-p_2)}^{\min([\frac{1}{2}(p_1+q_1)]-p_2,\, p_1-p_2)}$$

$$\times \sum_{n=\max(p_3,\, p_2+q_2,\, 2(p_2+q_2)-q_1)}^{\min(2(p_2+q_2),\, p_1)} 2\pi a_{p_2}^{p_1+q_1}\, b_{q_2}^{p_1+q_1-2p_2}\, d_{p_1,q_1}^{r,s}\, e_{p_3,q_3}^{r,p_1,n,n-p_2-q_2}\, f_t^{s,q_1+n-2(p_2+q_2)}$$

$$\times\, g_n^{p_1,q_1,p_2+q_2} \int_0^{\pi/2} \mathsf{P}_{p_1+q_1-2p_2}(\cos\phi)\, \mathsf{S}(\theta,\ w)\, \sin\theta\, d\theta .$$

Since $\phi = \pi - 2\theta$, $B_{p_3,q_3,t}^{r,s}$ is a function of the relation speed w. In case of Maxwellian molecules, S being a function of θ alone, $B_{p_3,q_3,t}^{r,s}$ is a constant. Substituting (35)-(38) into (34) and using (4), (27), (28) and (30), we derive the following result

$$\mathrm{B}Y_{2r|s}(\boldsymbol{c}) = \sum_{p_3=0}^{\min(r+s,\,r)} \sum_{q_3=0}^{r-p_3} \sum_{t=0}^{s} B^{r,s}_{p_3,q_3,t}\, c^{2q_3} c_*^{2(r-p_3-q_3)} (\boldsymbol{c}\cdot\boldsymbol{c})^{p_3} \tag{40}$$

$$\times\, c_{\{k_1} \cdots c_{k_t} c_{*k_{t+1}} \cdots c_{*k_s\}} - 2\pi \int_0^{\pi/2} \mathrm{S}(\theta,\ w) \sin\theta\, d\theta [Y_{2r|s}(\boldsymbol{c}) + Y_{2r|s}(\boldsymbol{c}_*)] .$$

In order to obtain the final result, we need to expand $c^{2q_3} c_*^{2(r-p_3-q_3)} (\boldsymbol{c}\cdot\boldsymbol{c}_*)^{p_3}$ $c_{\{k_1 \ldots k_t} c_{k_{t+1} \ldots k_s\}}$ in a bilinear combination of polynomials in the components $\boldsymbol{c}$ and $\boldsymbol{c}_*$ defined by Ikenberry. Setting $l_i = k_i$ $(1 \leqslant i \leqslant t)$, $l_{i+t} = m_i$ $(1 \leqslant i \leqslant p_3)$, in virtue of (9) and (23) we find that

$$(\boldsymbol{c}\cdot\boldsymbol{c})^{p_3} c_{k_1} \ldots c_{k_t} = c_{*m_1} \ldots c_{*mp_3} c_{k_1} \ldots c_{k_t} c_{m_1} \ldots c_{mp_3} \tag{41}$$

$$= c_{*m_1} \ldots c_{*mp_3} \sum_{a_1=0}^{\min([\frac12(t+p_3)],\,p_3)} a^{t+p_3}_{a_1} c^{2a_1} Y_{(l_1 \ldots l_{t+p_3-2a_1}}(\boldsymbol{c})\, \delta_{l_{t+p_3-2a_1+1} l_{t+p_3-2a_1+2}} \cdots \delta_{l_{t+p_3-1} l_{t+p_3})} + \mathrm{R}$$

$$= \sum_{a_1=0}^{\min([\frac12(t+p_3)],\,p_3)} \sum_{a_2=\max(a_1,\,2a_1-t)}^{\min(2a_1,\,p_3)} a^{t+p_3}_{a_1} g^{p_3,t,a_1}_{a_2} c^{2a_1} c_*^{2(a_2-a_1)} c_{*m_1} \ldots c_{*m_{p_3-a_2}}$$

$$\times\, Y_{m_1 \ldots m_{p_3-a_2}(k_1 \ldots k_{t+a_2-2a_1}}(\boldsymbol{c})\, c_{*k_{t+a_2-2a_1+1}} \ldots c_{*k_t)} + \mathrm{R} .$$

Similarly, using (9) and (25), we obtain

$$Y_{m_1 \ldots m_{p_3-a_2} k_1 - k_{t+a_2-2a_1}}(\boldsymbol{c})\, c_{*k_{t+a_2-2a_1+1}} \ldots c_{*k_t} c_{*k_{t+1}} \ldots c_{*k_s} c_{*m_1} \ldots c_{*mp_3-a_2} \tag{42}$$

$$= \sum_{a_3=0}^{\min(s+2a_1-t-a_3,\,p_3-a_2)} a^{p_3+s+2a_1-t-2a_2}_{a_3} g^{p_3-a_2,\,s+2a_1-t-a_2,\,a_3}_{a_3} c_*^{2a_3}$$

$$\times\, Y_{m_1 \ldots m_{p_3-a_2-a_3} k_1 \ldots k_{t+a_2-2a_1} k_{t+a_2-2a_1+1} \ldots k_{t+a_2+a_3-2a_1}}(\boldsymbol{c})\, Y_{k_t+a_2+a_3-2a_1+1 \ldots k_s m_1 \ldots m_{p_3-a_2-a_3}}(\boldsymbol{c}_*) + \mathrm{R} .$$

Making use of (41), (42), (27)-(31) and (8), we easily see that

$$c^{2q_3} c_*^{2(r-p_3-q_3)} (\boldsymbol{c}\cdot\boldsymbol{c}_*)^{p_3} c_{\{k_1 \ldots k_t} c_{*k_{t+1}} \ldots c_{*k_s\}} \tag{43}$$

$$= \sum_{a_1=0}^{\min([\frac12(t+p_3)],\,p_3)} \sum_{a_2=\max(a_1,\,2a_1-t)}^{\min(2a_1,\,a_3)} \sum_{a_3=0}^{\min(s+2a_1-t-a_2,\,p_3-a_2)} a^{t+p_3}_{a_1} a^{p_3+s+2a_1-t-2a_2}_{a_3}$$

$$\times\, g_{a_2}^{p_3,t,a_1}\, g_{a_3}^{p_3-a_2,s+2a_1-t-a_2,a_3}\, Y_{2(q_3+a_1)|m_1\dots m_{p_3-a_2-a_3}k_1\dots k_{t+a_2+a_3-2a_1}}(\boldsymbol{c})$$

$$\times\, Y_{2(r-q_3-a_1-p_3+a_2+a_3)|k_{t+a_2+a_3-2a_1+1}\dots k_s m_1\dots m_{p_3-a_2-a_3}}(\boldsymbol{c}_*)\,.$$

Exchanging the order of summation, we get

$$(44)\qquad \sum_{p_3=0}^{\min(2[\frac12(r+s)],\,r)}\ \sum_{q_3=0}^{r-p_3}\ \sum_{t=0}^{s}\ \sum_{a_1=0}^{\min([\frac12(t+p_3)],\,p_3)}\ \sum_{a_2=\max(a_1,\,2a_1-t)}^{\min(2a_1,\,p_3)}\ \sum_{a_3=0}^{\min(s+2a_1-t-a_3,\,p_3-a_2)}$$

$$= \sum_{a_1=0}^{\min([\frac12(r+s)],\,r)}\ \sum_{q_3=0}^{\min(r-a_1,\,s+r-2a_1)}\ \sum_{a_2=\max(a_1,\,2a_1-s)}^{\min(2a_1,\,r-q_3,\,2[\frac12(r+s)])}\ \sum_{a_3=0}^{\min(s,\,r-q_3-a_2,\,2[\frac12(r+s)]-a_2)}\ \sum_{p_3=a_2+a_3}^{\min(2[\frac12(r+s)],\,r-q_3)}\ \sum_{t=2a_1-a_2}^{\min(s,\,s+2a_1-a_2-a_3)}\,.$$

Let $\sum^*$ denote the summation operator defined by the right-hand side of (44). Placing (43) into (40) and using (44), we have

$$(45)\qquad \mathsf{B}Y_{2r|s} = \sum{}^{*} B^{r,s}_{p_3,q_3,t,a_1,a_2,a_3}\, Y_{2(q_3+a_1)|m_1\dots m_{p_3-a_2-a_3}k_1\dots k_{t+a_2+a_3-2a_1}}(\boldsymbol{c})$$

$$\times\, Y_{2(r-q_3-a_1-p_3+a_2+a_3)|k_{t+a_2+a_3-2a_1+1}\dots k_s m_1\dots m_{p_3-a_2-a_3}}(\boldsymbol{c}_*)$$

$$-\,2\pi\int_0^{\pi/2} \mathsf{S}(\theta,\ w)\,\sin\theta\, d\theta\,(Y_{2r|s}(\boldsymbol{c}) + Y_{2r|s}(\boldsymbol{c}_*))$$

in which

$$(46)\qquad B^{r,s}_{p_3,q_3,t,a_1,a_2,a_3} = B^{r,s}_{p_3,q_3,t}\, a_{a_1}^{t+p_3}\, a_{a_3}^{p_3+s+2a_1-t-2a_2}\, g_{a_2}^{p_3,t,a_1}\, g_{a_3}^{p_3-a_2,s+2a_1-t-a_2,a_3}\,.$$

Setting $q_3 + a_1 = l$, $p_3 - a_2 - a_3 = k$, and $t + a_2 + a_3 - 2a_1 = p$, and exchanging the order of summations we find that

$$(47)\qquad \mathsf{B}Y_{2r|s} = \sum_{k=0}^{\min(2[\frac12(r+s)],\,r)}\ \sum_{l=0}^{r-k}\ \sum_{p=0}^{s} \mathsf{m}A^{r,s}_{k,l,p}$$

$$\times\, Y_{2l|m_1\dots m_k k_1\dots k_p}(\boldsymbol{c})\, Y_{2(r-l-k)|k_{p+1}\dots k_s m_1\dots m_k}(\boldsymbol{c}_*)$$

$$-\,2\pi\int_0^{\pi/2} \mathsf{S}(\theta,\ w)\,\sin\theta\, d\theta\,[Y_{2r|s}(\boldsymbol{c}) + Y_{2r|s}(\boldsymbol{c}_*)]$$

in which

$$A^{r,s}_{k,l,p} = \frac{1}{\mathsf{m}} \sum_{a_1=0}^{\min([\frac{1}{2}(r+s)],\, l,\, 2[\frac{1}{2}(r+s)]-k,\, s+r-l-k-p,\, [\frac{1}{2}(s+r)]+[\frac{1}{2}(s-p-k)])} \tag{48}$$

$$\times \sum_{a_2=\max(a_1,\, 2a_1-s,\, p+2a_1-2s)}^{\min(2a_1,\, 2[\frac{1}{2}(r+s)]-k,\, r+a_1-l-k)} \; \sum_{a_3=\max(0,\, p+2a_1-s-a_2)}^{\min(p,\, r+a_1-l-k-a_2,\, 2[\frac{1}{2}(r+s)]-a_2-k)} B^{r,s}_{k+a_2+a_3,l-a_1,p+2a_1-a_2-a_3,a_1,a_2,a_3}\,.$$

It follows from (4) that $\mathsf{B}Y_{2r|s}$ is a symmetric function in the components of $\boldsymbol{c}$ and $\boldsymbol{c}_*$, so $A^{r,s}_{k,l,p} = A^{r,s}_{k,r-k-l,s-p}$. For Maxwellian molecules, $\mathsf{S}(\theta,\ w) = \mathsf{S}(\theta)$, and $A^{r,s}_{k,l,p}$ is independent of the relative speed w. It follows from (47) that $\mathsf{B}Y_{2r|s}$ is a polynomial in the components of $\boldsymbol{c}$ and $\boldsymbol{c}_*$ and its linear part is

$$(\mathsf{m}A^{r,s}_{0,0,0} - 2\pi \int_0^{2\pi} \mathsf{S}(\theta) \sin\theta \, d\theta)(Y_{2r|s}(\boldsymbol{c}) + Y_{2r|s}(\boldsymbol{c}_*))\,.$$

Placing (47) into (3) and using (10) and the Ikenberry's theorem, we obtain the following result

Theorem. *Under the assumptions leading to Ikenberry's theorem*

$$\mathsf{m}\bar{\mathsf{C}}Y_{2r|s} = -\,\mathsf{C}_{2r|s}\,\mathsf{P}_{2r|s} + \mathsf{Q}_{2r|s} \qquad 2r+s \geqslant 0 \tag{49}$$

in which

$$\mathsf{Q}_{2r|s} = \sum_{k=0}^{\min(2[\frac{1}{2}(r+s)],\, r)} \sum_{l=0}^{r-k} \sum_{p=0}^{s} \tfrac{1}{2} A^{r,s}_{k,l,p}[1 - \delta_{0k}(\delta_{0l}\delta_{0p} + \delta_{rl}\delta_{sp})] \tag{50}$$

$$\times\, \mathsf{P}_{2l|m_1\ldots m_k\{k_1\ldots k_p}\, \mathsf{P}_{2(r-k-l)|k_{p+1}\ldots k_s\}m_1\ldots m_k}$$

$A^{r,s}_{k,l,p}$ *being given* (48). *Using the symmetry of* (50), *we can write* (50) *also in the following way*

$$\mathsf{Q}_{2r|s} = \sum_{k=0}^{\min(2[\frac{1}{2}(r+s)],\, r)} \; \sum_{\substack{0\leqslant l\leqslant r-k,\ 0\leqslant p\leqslant s \\ 2l+p\geqslant \frac{s}{2}+r-k}} A^{r,s}_{k,l,p}[1 - \delta_{k0}\delta_{rl}\delta_{sp} - \tfrac{1}{2}\delta_{l,r-k-l}\delta_{p,s-p}] \tag{51}$$

$$\times\, \mathsf{P}_{2l|m_1\ldots m_k\{k_1\ldots k_p}\, \mathsf{P}_{2(r-l-k)|k_{p+1}\ldots k_s\}m_1\ldots m_k}\,.$$

We see from (48), (46) and (39) that the general expression $A^{r,s}_{k,l,p}$ is very complex, but in one special case $r=0$, the result is very simple

(52)
$$\mathrm{Q}_{0|s} = \sum_{p=0}^{s} \tfrac{1}{2} A^{0,s}_{0,0,p}(1-\delta_{0p}-\delta_{sp})\, \mathrm{P}_{0|\{k_1\ldots k_p}\, \mathrm{P}_{0|k_{p+1}\ldots k_s\}}$$

$$= \sum_{p=[\frac{s+1}{2}]}^{s-1} A^{0,s}_{0,0,p}\,(1-\tfrac{1}{2}\delta_{p,s-p})\, \mathrm{P}_{0|\{k_1\ldots k_p}\, \mathrm{P}_{0|k_{p+1}\ldots k_s\}}$$ in which

(53)
$$A^{0,s}_{0,0,p} = \frac{\pi}{2^{s-2}} \sum_{q=0}^{[\frac{1}{2}s]} \binom{s}{2q} f^{s,2q}_{p} \int_0^{\pi/2} \mathrm{P}_{2q}(\cos\phi)\, \mathrm{S}(\theta)\, \sin\theta\, d\theta$$

$f^{s,2q}_{p}$ being defined by (20).

The expression (53) is easily obtained from its definition. First, we know from (48), (46) and (39) that

$$A^{0,s}_{0,0,p} = B^{0,s}_{0,0,p,0,0,0} = B^{0,s}_{0,0,p}\, a^p_0\, a^{s-p}_0\, g^{0,p,0}_0\, g^{0,s-p,0}_0$$

$$= \sum_{q_1=0}^{s} 2\pi a^{q_1}_0\, b^{q_1}_0\, d^{0,s}_{0,q_1}\, e^{0,0,0,0,0}_{0,0}\, f^{s,q_1}_{p}\, g^{0,q_1,0}_0\, a^p_0\, a^{s-p}_0\, g^{0,p,0}_0\, g^{0,s-p,0}_0 \int_0^{\pi/2} \mathrm{P}_{q_1}(\cos\phi)\, \mathrm{S}(\theta)\, \sin\theta\, d\theta\,.$$

Next, by (6), (10), (16), (18) and (20), we obtain

$$A^{0,s}_{0,0,p} = \sum_{q_1=0}^{s} 2\pi \frac{1}{2^s} \binom{s}{q_1} (1+(-1)^{q_1})\, f^{s,q_1}_{p} \int_0^{\pi/2} \mathrm{P}_{q_1}(\cos\phi)\, \mathrm{S}(\theta)\, \sin\theta\, d\theta$$

$$= \sum_{q=0}^{[\frac{1}{2}s]} \frac{\pi}{2^{s-2}} \binom{s}{2q} f^{s,2q}_{p} \int_0^{\pi/2} \mathrm{P}_{2q}(\cos\phi)\, \mathrm{S}(\theta)\, \sin\theta\, d\theta\,.$$

This is precisely the expression (53).

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