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A fuzzy alternative set theory (**)

Introduction

In this paper we sketch the development of the theory FAST (Fuzzy Alternative Set Theory) proposed in [6] to which we refer for details.

Two axiomatizations of Fuzzy Sets, [1] and [10] paralleling ZF's theory, try to axiomatize Fuzzy Sets of [11] as near as possible, but they do not overcome problems of Fuzzy Sets (see [3]). Objects of the everyday life are fuzzy (i.e. are structured as Fuzzy Sets with membership degrees and valuation objects) but seem to satisfy the following properties too:

(1) different objects can have different valuation objects;

(2) the valuation object is as fuzzy as the other objects (in particular it has membership degrees and a valuation object);

(3) there exists only one definition of union, intersection, complementation.

The axiomatizations proposed in [1], [10], do not satisfy these conditions since the valuation class is fixed (even if it is not explicitely written) and it is not so fuzzy (in some sense) like the other classes as [1] and [10] put two crisp comparison relations on degrees. Nor is (3) satisfied as they set only one of the possible definitions of union and intersection (the most natural though).

Assuming an axiom of construction set-operations would be uniquely determined by the meaning of propositional connectives. But, if we want different valuation objects it seems impossible to assume extensionality on every pair of objects: that is we cannot say that $X = Y$ in case they are equiextensional. Indeed writing $\mathrm{val}(X)$ for the valuation object of X, defining $X = Y$ we should

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also define at the same time when $\mathrm{val}(X) = \mathrm{val}(Y)$, $\mathrm{val}(\mathrm{val}(X)) = \mathrm{val}(\mathrm{val}(Y))$ and so on. This problem is overcome with a suitable abstraction operator axiomatized in the theory.

Axioms on sets (clear objects) are given in such a way that sets satisfy axioms of the Alternative Set Theory (AST) (see $[8]_1$ or [9] or [5]).

AST try to axiomatize objects of the everyday life that are finite from the classical point of view. Thus a (classical) axiom of infinity is not assumed. In AST sets are objects with well-defined boundaries while classes have ill-defined boundaries, and proper classes can be included in sets (see $[8]_1$ or [9] for examples). This observation suggested to look for an axiomatization unifying these two kinds of vagueness.

Other peculiar aspects of AST (and FAST) are that only two infinite cardinalities (as defined inside the theory) are assumed, and particular classes of classes can be «translated» (coded) inside the core of the theory in which mathematics can be done (Universe of Sets).

The theory of $[8]_2$ is an axiomatization of the theory exposed in [9] in a semiformal way, but reflects only the core of the theory of [9] and the membership relation between classes can be assumed only using a heavy formalization.

Our theory allows membership between classes and other kinds of objects can be considered (as in [9]). Some axioms of AST are generalized: the axioms of induction, of existence of sets and of prolungation. On the other hand it seems not suitable to generalize other axioms (like those of two cardinalities and of extensional coding) in view of the structure of objects. We noted that extensionality must not be assumed everywhere, but we can assume it on definable objects and on some other objects of the theory: classes (of our theory). FAST is in some way intensional.

The theory

1 - Metadefinition. (a) The language we use is $\mathscr{L} = (\in, \equiv, \{.:.\})$ where $\in$ is a quaternary predicate representing membership, $\equiv$ is the equality predicate and $\{.:.\}$ is the abstraction operator (see [4]).

(b) One should read $\in(A, B, C, D)$ as «A belongs to B with (membership) degree C w.r.t. (the valuation object) D».

(c) The entities considered in our theory are «objects» and are written with the first uppercase latin letters of the alphabet, A, B, C, ... possibly indexed.

From now on we shall use informal language mixed with formulas of $\mathscr{L}$ to enhance readability.

2 - The first axioms of the theory give the structure of the objects.

Axiom FAST 1. (Uniqueness of the valuation object).

$$(\forall B, D, D')(\forall A, A', C, C')[(\in(A, B, C, D) \wedge \in(A', B, C', D')) \rightarrow D \equiv D'].$$

Axiom FAST 2. (Uniqueness of the membership degree).

$$(\forall A, B, C, C')(\forall D, D')[(\in(A, B, C\ D) \wedge \in(A, B, C', D')) \rightarrow C \equiv C'].$$

Axiom FAST 3. (Fuzziness of the valuation object).

$$(\forall C, D)[((\exists A, B) \in(A, B, C, D)) \rightarrow ((\exists C', D') \in(C, D, C', D'))].$$

3 - Def. We say that A is an *element* and write $\mathrm{El}(A)$ if $(\exists B, C, D) \in(A, B, C, D)$. When $\in(A, B, C, D)$ we say that C is the *membership degree* of A in B, write $\mu_B(A) \equiv C$, D is the *valuation object* of B, write $\mathrm{val}(B) \equiv D$.

Elements are for our theory what sets are for the Kelley-Morse theory. With the definition above we can give an intuitive interpretation of the axioms:

FAST 1. An object that is not empty has one and only one valuation object.

FAST 2. An element of the object B belongs to it with exactly one membership degree.

FAST 3. The valuation object has its own valuation object.

So, if $\in(A, B, C, D)$ then $\mathrm{El}(C)$.

4 - Recall that for a given formula $\varphi(A)$ the object obtained by the application of the abstraction operator to $\varphi(A)$ is written $\{A: \varphi(A)\}$.

Axiom FAST 4. (Construction).
Given a formula $\varphi(A)$ is an axiom

$$(\forall A)((\exists C, D) \in(A, \{B: \varphi(B)\}, C, D) \leftrightarrow (\mathrm{El}(A) \wedge \varphi(A))).$$

Usual set-operations are ensured by the following

Axiom FAST 5. (Extensionality on definable objects).
Given the formulas $\varphi(A)$ and $\psi(A)$

$$(\forall A)[\mathrm{El}(A) \rightarrow (\varphi(A) \leftrightarrow \psi(A))] \rightarrow \{B: \varphi(B)\} \equiv \{B: \psi(B)\}.$$

Convention. From now on the last uppercase latin letters of the alphabet (W, X, ...) are names for variables relativized to elements; we write $\{X: \varphi(X)\}$ for $\{A: \varphi(A)\}$.

We shall define sets as the objects with one possible membership degree. We need then the following notion of equiextensionality.

Some useful notations are the following:

(a) $\emptyset$ is the object $\{X: \sim(X \equiv X)\}$.

(b) If $X_1, \ldots, X_n$ are elements, then $\{X_1, \ldots, X_n\}$ is shorthand for $\{Y: Y \equiv X_1 \vee \ldots \vee Y \equiv X_n\}$.

(c) El is the object $\{X: X \equiv X\}$.

5 - Def. (a) A is a *set*, write $\boldsymbol{W}(A)$, if $(\mathrm{El}(A) \wedge (A \equiv \emptyset \vee (\exists X)\ \mathrm{val}(A) \equiv \{X\}))$.

(b) $\boldsymbol{W}$ is the object $\{X: \boldsymbol{W}(X)\}$.

(c) A and B are *equiestensional*, write $A = B$, if $(\forall X)$ $[(\exists C, D) \in (X, A, C, D) \leftrightarrow (\exists C', D') \in (X, B, C', D')]$.

Of course $\emptyset$ will be a set with no elements and it may not be an element up to now; $=$ is an equivalence relation. We can prove that $(\forall A)$ $A = \{X: (\exists C, D) \in (X, A, C, D)\}$, but we cannot strengthen it to equality $\equiv$. By axiom 5 on definable objects one has

$$\{X: \varphi(X)\} = \{X: \psi(X)\} \leftrightarrow \{X: \varphi(X)\} \equiv \{X: \psi(X)\}.$$

By means of the abstraction operator it is possible to define the operations of union, intersection and complementation as in classical set-theory. We can prove as usual that these operations satisfy the same properties of classical sets. For example: $A \cap -A \equiv \emptyset$ and $A \cup -A \equiv \mathrm{El}$ (for this aspect see [7]). But some properties hold only up to equiextensionality:

idempotency	$A \cup A = A,\ A \cap A = A.$
identity	$A \cup \emptyset = A,\ A \cap \mathrm{El} = A.$
absorption	$A \cup (A \cap B) = A,\ A \cap (A \cup B) = A.$
involution	$--A = A.$

Of course if $A \equiv \{X: \varphi(X)\}$ then equality holds in the formulas above (compare with the properties satisfied by F-sets in [2]).

6 - With the following axiom we begin the list of axioms on sets that reflect the Alternative point of view.

Axiom FAST 6. (Existence of set).

$$\boldsymbol{W}(\emptyset) \wedge (\forall A, X)(\boldsymbol{W}(A) \rightarrow \boldsymbol{W}(A \cup \{X\})).$$

Therefore if X and Y are elements then $\{X\}$, $\{X, Y\}$, $\langle X, Y\rangle$ are sets.

7 - Def. (a) An element X is an *urelement*, or *atom* and write $\mathrm{Ur}(X)$ if $X = \emptyset$.

(b) Ur is the object $\{X: \mathrm{ur}(X)\}$.

(c) $A \subseteq B$ is the formula $(\forall X)\ [((\exists C, D) \in (X, A, C, D)) \rightarrow ((\exists C', D') \in (X, B, C', D'))]$.

(d) A is a *class*, $\mathrm{Clas}(A)$, if $(\sim \mathrm{Ur}(A) \wedge (\boldsymbol{W}(A) \vee A \subseteq \boldsymbol{W} \cup \mathrm{Ur}))$.

(e) $\mathrm{Pr}(A)$ is the formula $\mathrm{Clas}(A) \wedge \sim \boldsymbol{W}(A)$.

Convention. From now on lowercase letters are names for variables which are relativized to sets.

Note that $\boldsymbol{W}$ and Ur are proper classes but El is an object and need not be a class. Note that for every A, B: $(\mathrm{Ur}(A) \rightarrow A \subseteq B)$, $((A \subseteq B \wedge B \subseteq A) \rightarrow A = B)$, $((A \cup B) = B \leftrightarrow A \subseteq B)$.

8 - We can assume extensionality on a wider collection of objects.

Axiom FAST 7. (Extensionality on classes).

$$(\forall A, B)((\mathrm{Clas}(A) \wedge \mathrm{Clas}(B)) \rightarrow (A \equiv B \leftrightarrow A = B)).$$

In this theory Fuzzy Sets à la Zadeh can be defined as «approximations of real objects» justifying, with the definition, the presence in naive Fuzzy Sets Theory of several notions of set operations as union, bold union, probabilistic sum (see [2]).

Sets are those objects on which the membership relation can be reduced to appear as a binary relation. It may be useful in certain cases to employ a set-theoretical like notation by adopting the following

Convention. Let $A \in B$ be the formula $(\exists C, D) \in (A, B, C, D)$.

9 - Def. The following are definable objects:

(a) $\boldsymbol{P}(A) \equiv \{X \colon X \subseteq A\}$ is the *power* of A.
(b) $\boldsymbol{P}_s(A) \equiv \{x \colon x \subseteq A\}$ is the *set-power* of A.
(c) $U_s(A) \equiv \{X \colon (\exists y)(y \in A \land X \in y)\}$ is the *set-union* of A.
(d) $U(A) \equiv \{X \colon (\exists Y)(Y \in A \land X \in Y)\}$ is the *union* of A.

Note the differences between (a) and (b), and (c) and (d).

Now we arrive at the notion of a set-formula which is essential for the axiom of induction.

10 - Def. (a) Define recursively

(I) (a) $X \in x$, (b) $y \in x$, (c) $X \equiv x$, (d) $X \equiv Y$, (e) $y \equiv x$, are *strong set-formulas of s-height O*; then suppose that strong set-formulas of s-height n have been defined for every $n < m$, then

(II) if $\varphi(X)$, $\psi(X)$ and $\nu(X)$ are *strong-set-formulas* of s-height respectively p, q, r, with p, q, r such that $p + q = m - 1$, $r = m - 1$, then $\sim \nu(X)$, $\varphi(X) \land \psi(X)$, $(\exists X)\, \nu(X)$, $(\exists x)\, \nu(x)$ are *set-formulas of s-height m.*

(b) A formula φ is a *set-formula* if there exists an n and a strong set-formula ψ of a s-height n such that φ and ψ are logically equivalent.

(c) If $B \equiv \{X \colon \varphi(X)\}$ and φ is a set-formula we say that B is *set-definable.*

For example $x \subseteq \boldsymbol{W}$ is a set-formula.

Axiom FAST 8. (Induction).
If $\varphi(X)$ is a set-formula then is an axiom

$$[\varphi(\emptyset) \land (\forall x)(\forall X)(\varphi(x) \rightarrow \varphi(x \cup \{X\}))] \rightarrow (\forall x)\, \varphi(x).$$

By this axiom the following theorems can be proved.

11 - Theorem. (a) (Replacement) *If* φ *is a set-formula*

$$(\forall X)(\exists! Y)\varphi(X, Y) \to (\forall a) W(\{Y: (\exists X \in a)\varphi(X, Y)\}).$$

(b) (Separation) *If* φ *is a set-formula*

$$(\forall x)(\exists z)(\forall X)(X \in z \leftrightarrow (X \in x \wedge \varphi(X))) \quad \text{that is} \quad W(\{X: X \in x \wedge \varphi(X)\}).$$

Moreover we can prove by induction that $x \cup y$, $U_s(x)$, $P_s(x)$ are sets, and that for every set x there exists a relation r that is a set and satisfies the maximum and minimum principle for every non-empty subset of x.

We define then natural numbers from which we will obtain the Universe of Sets by recursion.

Now we can define when A is a relation, Rel(A), or a function, Fnc(A), as in KM. If R is a relation we write $R(X, Y)$ instead of $\langle X, Y\rangle \in R$. Then define

12 - Def. (a) $\underline{E} \equiv \{X: (\exists Y, Z)\ X \equiv \langle Y, Z\rangle \wedge (Y \in Z \vee Y \equiv Z)\}$.

(b) OL(R, B) is for $\mathrm{Rel}(R) \wedge (\forall Y, Z)((Y \in B \wedge Z \in B) \to (R(Y, Z) \vee Y \equiv Z \vee R(Z, Y)))$.

(c) Tr(A) is for $(\forall Y, Z)((Z \in Y \wedge Y \in A) \to Z \in A)$.

(d) If x is a set, reg(x) is for $(\forall z \subseteq x)[((\exists t)\, t \in z) \to (\exists y \in z)\, y \cap z \equiv \emptyset]$.

(e) $N \equiv \{x: \mathrm{Tr}(x) \wedge \mathrm{OL}(\underline{E}, x) \wedge x \subseteq W \wedge \mathrm{reg}(x)\}$, if $x \in N$ we say that x is a *natural number*.

(f) $\leqslant \equiv \underline{E} \cap N^2$.

Note that $(x \in N \to (\{x_1, \ldots, x_n\} \subseteq x \to \sim (x_1 \in x_2 \wedge \ldots \wedge x_{n-1} \in x_n \wedge x_n \in x_1))$.

Properties of natural numbers can be proved (see [9]), in particular

13 - Theorem. *If* A *is a set-definable object and* $A \cap N \neq \emptyset$, *then there exists the first element of* $A \cap N$ *in the relation* $\leqslant$.

We introduce the notions of semiset and finite object.

14 - Def. (a) Sem(A) is for $(\exists y)\ A \subseteq y$ and we say that A is a *semiset*; a semiset is *proper* if it is not a set.

(b) A is *finite*. FIN(A), iff $(\forall B)((B \subseteq A \wedge \sim \mathrm{Ur}(B)) \to W(B))$.

(c) Given A, if there exists R such that:

(1) $\mathrm{OL}(R, A)$, (2) $\sim \mathrm{Ur}(A) \wedge \sim \mathrm{FIN}(A)$, (3) $(\forall Z \in A)\, \mathrm{FIN}(\{Y: Y \in A \wedge R(Y, Z)\})$, we say that the pair A, R is an *ordering of type* ω, $O\omega(R, A)$ and that A is *countable* ($\mathrm{COUNT}(A)$).

(d) $\mathrm{UNCOUNT}(A)$ is for $\sim \mathrm{Ur}(A) \wedge \sim \mathrm{FIN}(A) \wedge \sim \mathrm{COUNT}(A)$ and we say that A is *uncountable.*

(e) $A \cong B$ is for $(\exists F)$ («F is an injection» $\wedge \mathrm{Dom}(F) \equiv A \wedge \mathrm{Rng}(F) \equiv B$).

If A and B are semisets then $A \cap B$, $A \cup B$, $A - B$, $U_s(A)$ are semisets, and if A is a set-definable semiset, then is a set.

Some usual properties of finite objects can be proved: if A and B are finite then, $W(A)$, $\mathrm{FIN}(\{A\})$, $\mathrm{FIN}(A \cup B)$, $\mathrm{FIN}(P_s(A))$ and so on.

15 - Now we assume (note that $\mathrm{Fnc}(F) \to F \subseteq W$)

Axiom FAST 9. (Prolungation).

$$(\forall F)[(\mathrm{COUNT}(F) \wedge \mathrm{Fnc}(F)) \to (\exists f)(\mathrm{Fnc}(f) \wedge \mathrm{F} \subseteq \mathrm{f})].$$

By prolungation we can prove that $\mathrm{FN} \equiv \{x: x \in N \wedge \mathrm{FIN}(x)\}$ is countable, is a proper semiset, if A is countable then $A \cong \mathrm{FN}$ and there exists a set a such that $A \subseteq a$, that there exists an infinite set and an infinite natural number.

Therefore we can prove the following theorem (where we use the notation $F \upharpoonright A$ for the *restriction of* F to A)

16 - Theorem. (Recursion). *If G is a set-definable function and* $\mathrm{Dom}(G) \equiv W$, *then there exists a unique set-definable function F such that* $\mathrm{Dom}(F) \equiv N \wedge (\forall x \in N)\ F(x) \equiv G(F \upharpoonright x)$.

With this we can define the Universe of sets.

17 - Def. (a) P is the unique set-definable function with $\mathrm{Dom}(P) \equiv N$, obtained by recursion from $H \equiv \{\langle x, y\rangle: y \equiv P_s(x)\}$.

(b) $V \equiv U(\mathrm{Rng}(P))$ is the *Universe of sets.*

(c) $(\forall a)$ $a \in V$, $rk(a)$ (the *rank* of a) is for the minimum of $\{x: x \in N \wedge a \in P(x)\}$.

(d) If $A \subseteq V$ then we say that A is a *class of the extended universe.*

Note that V is set-definable and that in ZF the rank of a set x is the minimum of $\{y: y \in N \wedge x \in P(y+1)\}$ w.r.t. $\leqslant$.

Using the function P we can see that $N \subsetneq V$, $V \subsetneq W$, $\mathrm{Tr}(V)$, $(x \subseteq V \rightarrow x \in V)$, $(A \subseteq V \rightarrow \mathrm{Clas}(A))$.

18 - Def. A *V-formula* is a formula in which every variable (free or bounded) is restricted to V.

Obviously a V-formula is a set-formula. By this we can prove for example. Regularity on V. If $\varphi(x)$ is a V-formula

$$(\exists x)\,\varphi(x) \rightarrow (\exists x)(\varphi(x) \wedge (\forall y \in x)\ \sim \varphi(y))$$

analogously we prove that V satisfies the axioms of [9].

Note that on W there exists no form of regularity while V (which is obtained from W) is regular.

19 - Now we assume

Axiom FAST 10. (Two cardinalities).

$$(\forall A, B \subseteq V)((\mathrm{UNCOUNT}(A) \wedge \mathrm{UNCOUNT}(B)) \rightarrow A \cong B)$$

It seems interesting to assume

Axiom FAST 11. (Elementarity of the classes of the extended universe).

$$(\forall A)(A \subseteq V \rightarrow \mathrm{El}(A))\,.$$

With axiom 11 we can collect together every subclass of V and give

20 - Def. $\mathcal{U} \equiv \{X\colon X \subseteq V\}$ is the *extended universe*.

21 - Def. (a) Given K and S such that $K \in \mathcal{U} \wedge \mathrm{Fnc}(S) \wedge \mathrm{Dom}(S) \subseteq V \wedge \mathrm{Dom}(S) \supseteq K \wedge \mathrm{Cod}(S) \subseteq \mathcal{U}$, then we say that K, S is a *coding pair*, $\mathrm{CC}(K, S)$.

(b) If A is such that $A \subseteq \mathcal{U}$, and $\mathrm{CC}(K, S)$, then K and S *code* A iff $(\forall C)(C \in A \leftrightarrow (\exists y)\ (y \in K \wedge C \equiv S(y)))$, in simbol $\mathrm{CD}(K, S/A)$.

(c) If $A \subseteq \mathcal{U}$, A is *codable* iff $(\exists K, S)\ \mathrm{CD}(K, S/A)$.

(d) The coding pair K, S is *extensional*, $\mathrm{CE}(K, S)$, if S is injective.

Axiom FAST 12. (Extensional coding).

$$(\forall A)(\exists K, S)(\text{CD}(K, S/A) \rightarrow (\exists K', S')(\text{CE}(K', S') \wedge \text{CD}(K', S'/A))).$$

The well ordering of V implies the axiom of extensional coding which can be proved like in [9].

FAST is the theory collecting axioms from 1 to 12.

It can be proved then that the theory $\text{FAST}^- = \text{FAST} - \text{FAST } 11$, with the necessary changes in the definition of coding pair and FAST 12, is consistent with ZF and that the theory AST of $[8]_2$ has an interpretation in it (see [**6**]): indeed a model of the theory AST of $[8]_2$ is a model of FAST^-.

In a future paper we intend to show that FAST is consistent with ZF.

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Sommario

In questo articolo viene mostrato, in rapido sunto, lo sviluppo della teoria FAST *(Fuzzy Alternative Set Theory). Questa teoria assiomatizza oggetti che presentano sia caratteri propri di Fuzzy Sets, sia caratteri della Teoria Alternativa degli Insiemi* (AST)*: tali caratteri vengono introdotti e brevemente discussi. La teoria proposta generalizza le versioni formali finora proposte sia dei Fuzzy Sets sia di* AST.

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