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On the normal index of maximal subgroups of a finite group (**)

Introduction

Let M be a maximal subgroup of a finite group G. Let $F(M)$ denote the set of pairs $\{(H, X) | \langle M, H\rangle = G,\ H \trianglelefteq G,\ X \trianglelefteq G,\ X \underset{+}{\subseteq} H\}$. An element (H, K) in $F(M)$ is called a *minimal element* in $F(M)$ if H/K is a chief factor and $(T, V) \in F(M)$ with $T \subseteq H$ implies $T = H$. For a minimal element (H, K), $K = \text{core}\ M \cap H$ and if $N \trianglelefteq G$, $N \underset{+}{\subseteq} H$ then $N \subset K$. The order of the chief factor H/K is the normal index $\eta(G:M)$ of M.

The idea of normal index was introduced by Deskins [4]. Beidleman and Spencer [2], Mukherjee [6] and most recently Mukherjee and Battacharya $[7]_2$ have used this notion to characterize solvable, supersolvable, nilpotent, p-solvable, p-supersolvable and p-nilpotent groups. Here we consider specific classes of maximal subgroups and examine the influence of the normal indices of these on group structures.

1 - We shall use $Z_\infty(G)$ and $Q^*(G)$ to denote respectively the hypercenter and the hyperquasicenter of a group G. The quasicenter $Q(G) = Q_1(G)$ of G, is the characteristic subgroup generated by all cyclic quasinormal subgroups of G. The hyperquasicenter $Q^*(G)$ is the largest term of the chain of the subgroups $Q_0(G) = \{1\} \subseteq Q_1(G) \subseteq Q_2(G) \subseteq \dots$, where $\dfrac{Q_i(G)}{Q_{i-1}(G)} = Q(\dfrac{G}{Q_{i-1}(G)})$ for all $i > 0$.

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The following results on the hyperquasicenter will be relevant.

Lemma [9]. Let $\langle u\rangle$ be quasinormal in a group G. Then $\langle u^i\rangle$ is quasinormal for all integers i.

Theorem $[7]_2$. Let N be a normal subgroup of G and $N\subseteq Q^*(G)$ then

$$Q^*(\frac{G}{N})=\frac{Q^*(G)}{N}.$$

An analogous result is also valid when $Q^*(G)$ is replaced by $Z_\infty(G)$.

Theorem $[7]_2$. The hyperquasicenter $Q^*(G)$ of a group G is the largest supersolvably embedded subgroup of G.

Let G be a group and p be a given prime. Set

$$J_p=\{M\lessdot G|[G:M]_p=1\} \qquad R_p=\{M\lessdot G|[G:M]_p=1 \text{ and } [G:M] \text{ is composite}\}.$$

Define

$$\phi_p(G)=\cap\{M|M\in J_p\} \qquad S_p(G)=\cap\{M|M\in R_p(G)\}.$$

If $R_p=\phi$ then $S_p(G)=G$.

The normal index of a maximal subgroup M in G shall be denoted by $\eta(G:M)$. All other notations are standard.

2 - For the sake of completeness we include the following known proposition. However, the proof given in this paper is independent.

Proposition. *For a maximal subgroup M of a group G, the order of the chief factor H/K produced by each minimal element (H, K) in $F(M)$ is constant.*

Proof. Let (H_1, K_1) and (H_2, K_2) be two minimal elements in $F(M)$. If core $M\neq 1$ then consider $\frac{G}{N}$ where N is a minimal normal subgroup of G contained in

M. If $N \cap K_1 \neq 1$ then $N \subseteq K_1$ and $(\frac{H_1}{N}, \frac{K_1}{N})$ is a minimal element of $F(\frac{M}{N})$. If $N \cap K_1 = 1$ then $(\frac{H_1 N}{N}, \frac{K_1 N}{N})$ is a minimal element in $F(\frac{M}{N})$. In either case the order of the chief factor produced by the minimal element is $|\frac{H_1}{K_1}|$. By induction it now follows that $|\frac{H_1}{K_1}| = |\frac{H_2}{K_2}|$. Now assume core $M = 1$ so that $K_1 = K_2 = 1$ and H_1, H_2 are minimal normal subgroups of G, which therefore centralize each other. $C_G(H_1) \trianglelefteq G$ implies $C_G(H_1) \cap M \trianglelefteq M$ and therefore $C_G(H_1) \cap M \trianglelefteq G$ since $G = MH_1$. Thus $C_G(H_1) \cap M = 1$ and similarly $C_G(H_2) \cap M = 1$ so that $H_1 \cap M = H_2 \cap M = 1$. Now

$$H_1(H_1 H_2 \cap M) = H_1 H_2 = H_2(H_1 H_2 \cap M)$$

and this implies

$$H_2 \cong \frac{H_1 H_2}{H_1} \cong H_1 H_2 \cap M \cong \frac{H_1 H_2}{H_2} \cong H_1$$

and the proposition is proved completely.

Theorem 1. *Let p be the largest prime divisor of the order of a group G. If $\forall M \in J_p$, $F(M)$ contains a minimal element (H, K) such that $\frac{H}{K} \cap Q^*(\frac{G}{K}) \neq \bar{1}$ then G is a Sylow tower group of the supersolvable type.*

Proof. $\frac{H}{K} \cap Q^*(\frac{G}{K}) \neq \bar{1}$ implies $\frac{H}{K} \subseteq Q^*(\frac{G}{K})$ and $Q^*(\frac{G}{K})$ is supersolvably embedded in $\frac{G}{K}$ implies $\frac{H}{K}$ is of prime order. Therefore $\eta(G:M)$ is a prime. Since $[G:M]$ divides $\eta(G:M)$, it follows that $[G:M] = \eta(G:M) = a$ prime. Consequently $R_p = \phi$ and $G = S_p : G$ is solvable, and $P \trianglelefteq G$ [7]$_1$. Consider $\frac{G}{P}$ and $\forall \frac{Y}{P} < \frac{G}{P}$, $[\frac{G}{P} : \frac{Y}{P}]_p = 1$ so that $[G:Y]_p = 1$. Consequently $[G:Y] = a$ prime i.e. $[\frac{G}{P} : \frac{Y}{P}] = a$ prime. Therefore $\frac{G}{P}$ is supersolvable and the theorem is proved.

Corollary. *If $\eta(G:M)$ is square free $\forall M$ in J_p where p is the largest prime divisor of the order of G then G is a Sylow tower group of the supersolvable type.*

Proof. $\eta(G:M)$ is square free implies $\eta(G:M)=[G:M]=a$ prime. Hence for some minimal element (H, K) in $F(M)$, $\frac{H}{K}\cap Q^*(\frac{G}{K})\neq \bar{1}$ and therefore the assertion follows.

Mukherjee and Bhattacharya have shown $[7]_1$ that $S_p(G)\cap S_q(G)$ is supersolvable if p is the largest prime divisor of the order of G and q is any other divisor of $|G|$. In particular if $R_p=\phi=R_q$ then $G=S_p(G)=S_q(G)$ is supersolvable. This implies that a group G is supersolvable if and only if maximal subgroups whose indices do not contain p or q have prime indices. We use this to prove the following

Theorem 2. *Let p be the largest prime divisor of the order of a group G and q be any other divisor of $|G|$. Then G is supersolvable if and only if $\forall M$ in J_p or J_q, $F(M)$ contains a minimal element (H, K) such that* $\frac{H}{K}\cap Q^*(\frac{G}{K})\neq \bar{1}$.

Proof. If G is supersolvable then clearly the assertion is true.

Conversely, if for $M\in J_p$ or J_q, $\frac{H}{K}\cap Q^*(\frac{G}{K})\neq \bar{1}$ for some minimal element (H, K) in $F(M)$ then $\eta(G:M)=a$ prime and therefore $[G:M]=a$ prime. Hence $R_p=\phi$ and $R_q=\phi$. By the remark preceding the theorem it now follows that G is supersolvable.

If a group G is solvable and (H, K) is a minimal element in $F(M)$ then $Q^*(\frac{G}{K})$ need not be non trivial. In $G=A_4$, if $M=\langle X\rangle$, $|X|=3$ then $(V, 1)$ where V is the Klein-4 group is a minimal element in $F(\langle X\rangle)$ but $Q^*(\frac{A_4}{1})=\bar{1}$. If however $Q^*(\frac{G}{K})$ is non trivial for each minimal pair (H, K) in $F(M)$ then we obtain the following

Lemma. *If for each minimal pair (H, K) in $F(M)$, $Q^*(\frac{G}{K})\neq \bar{1}$ then M is of prime index in G.*

Proof. If core $M=1$ then $Q^*(G)\neq 1$ and there exists $\langle a\rangle \trianglelefteq G$, $|\langle a\rangle|=p$. Consequently $G=M\langle a\rangle$ and $[G:M]=p$. Suppose core $M\neq 1$ and let N be a minimal normal subgroup of G in core M. Then $\frac{M}{N}<\frac{G}{N}$ and if $(\frac{C}{N}, \frac{D}{N})$ is a minimal pair in $F(\frac{M}{N})$ then (C, D) is a minimal pair in $F(M)$ and therefore $Q^*(\frac{G}{D}) \neq \bar{1}$ which implies $Q^*(\frac{G/N}{D/N}) \neq \bar{1}$. By induction it now follows that M is of prime index in G.

Theorem 3. *Let p be the largest prime divisor of a group G. If $Q^*(\frac{G}{K}) \neq \bar{1}$ for each minimal element (H, K) in $F(M)$, $M \in J_p$ then G is a Sylow tower group of the supersolvable type.*

Proof. By the Lemma it follows that for $M \in J_p$, $[G:M] = a$ prime. Consequently $R_p = \phi$ and therefore $G = S_p(G)$. G is then solvable and since $p | G| = |S_p(G)|$ the Sylow p-subgroup of G is normal. By Corollary 6 (v. $[7]_2$) it now follows that G is a Sylow tower group of the supersolvable type.

Corollary. *G is supersolvable if and only if $\forall M$ in J_p or J_q where p is the largest prime divisor of $|G|$, and $p \neq q$, for a minimal element (H, K) in $F(M)$ $Q^*(\frac{G}{K}) \neq \bar{1}$.*

Proof. The assertion is certainly true if G is supersolvable. Conversely by the Lemma in above we get $R_p = \phi$ and $R_q = \phi$. This implies $G = S_p(G) = S_q(G)$ and by Theorem 11 (v. $[7]_1$) it follows that G is supersolvable.

We now investigate relationship between hypercenter and normal index of maximal subgroups whose indices are not divisible by some fixed prime.

Theorem 4. *If for some minimal element (H, K) in $F(M)$, $M \in J_p$ $\frac{H}{K} \cap Z_\infty(\frac{G}{K}) \neq \bar{1}$, then G is a Sylow tower group and is a p-group by a nilpotent p-group.*

Proof. $\frac{H}{K} \cap Z_\infty(\frac{G}{K}) \neq \bar{1}$ implies that $\frac{H}{K}$ is central and therefore $\frac{M}{K} \trianglelefteq \frac{G}{K}$ i.e. $M \trianglelefteq G$. If $p \in \mathrm{Syl}_p(M)$ and $P \ntrianglelefteq G$ then $G = XM$ where $X \lessdot G$ and $X \supset N_G(P)$. Now $[G:X]_p = 1$ and by the same argument as in above $X \trianglelefteq G$. Consider $P \subset X \cap M$. $P \lhd X \cap M \lhd G$ and this implies $P \trianglelefteq G$.

Consider $\frac{G}{P}$. For any maximal subgroup $\frac{Y}{P}$ of $\frac{G}{P}$, $[\frac{G}{P}:\frac{Y}{P}]_p = 1$ i.e. $[G:Y]_p = 1$ and once again $Y \trianglelefteq G$. Hence it now follows that $\frac{G}{P}$ is nilpotent and the theorem is proved.

Remark. It is not difficult to see that the converse is also true.

Corollary. *A group G is nilpotent if and only if for $\forall M \in J_p$ or J_q there exists a minimal element (H, K) in $F(M)$ such that* $\frac{H}{K} \cap Z_\infty(\frac{G}{K}) \neq \bar{1}$.

Proof. If P and Q are two Sylow subgroups corresponding to the primes p and q then $\frac{G}{P}$ and $\frac{G}{Q}$ are nilpotent by the theorem above and so $\frac{G}{P \cap Q} \cong G$ is nilpotent.

Lemma 2. *If for each minimal element (H, K) in $F(M)$, $Z_\infty(\frac{G}{K}) \neq \bar{1}$ then $M \trianglelefteq G$.*

Proof. If core $M = 1$ then $Z_\infty(G) \neq 1$ and clearly $M \trianglelefteq G$. Let N be a minimal normal subgroup of G in core M and consider $\frac{G}{N}$. If $(\frac{C}{N}, \frac{D}{N})$ is a minimal element in $F(\frac{M}{N})$ then (C, D) is a minimal element in $F(M)$ and therefore $Z_\infty(\frac{G}{D}) \neq \bar{1}$ i.e. $Z_\infty(\frac{G/N}{D/N}) \neq \bar{1}$. By induction it now follows that $M \trianglelefteq G$.

Theorem 5. *The Sylow p-subgroup p of a group G is normal and $\frac{G}{P}$ is nilpotent if and only if for each minimal element (H, K) in $F(M)$, $M \in J_p$ it follows that $Z_\infty(\frac{G}{K}) \neq \bar{1}$.*

Proof. If $\frac{G}{P}$ is nilpotent then $\frac{M}{P} \lessdot \frac{G}{P}$ $\forall M \in J_p$ and therefore $M \trianglelefteq G$. Hence (G, M) is a minimal pair in $F(M)$ and is in fact the only minimal pair.

Conversely, by the Lemma 2 above $M \in J_p$ it implies that $M \trianglelefteq G$. Arguing as in Theorem 4, it follows that $P \trianglelefteq G$, where P is a Sylow subgroup of G. Now consider $\frac{G}{P}$ and $\frac{X}{P} \lessdot \frac{G}{P}$ implies $X \in J_p$ and therefore $X \trianglelefteq G$. Hence $\frac{G}{P}$ is nilpotent and we are done.

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Summary

See Introduction.

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