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ALAN C. WILDE (*)

Properties of affine projections (**)

1 - Introduction

In the present paper, we discuss properties of affine projections in the frame of affine geometry. Let V be a linear space on which no topology is assigned. If p as usual denotes a linear projection operator in V, that is, $p(p(x)) = p(x)$ for all x in V, then $P(x) = a + p(x)$ is said to be an *affine projection* if $a + p(a + p(x)) = a + p(x)$.

Let p, q be linear projections in V which commute. We prove first (cfr. 3) that the sum of the ranges of the two affine projections $P(x) = a + p(x)$ and $Q(x) = b + q(x)$ is the range of an affine projection. We also prove (cfr. 4) that if the ranges of the two affine projections intersect, then their intersection is also the range of an affine projection. We further prove that the ranges of two affine projections intersect if and only if they commute under composition. These results apply e.g. to the problem we discussed in Wilde $[2]_1$ as to whether there are solutions x in V to a system of equations $p_i(x) = k_i$ $(i = 1, \ldots, n)$ where p_i are given linear, commuting projections and k_i is any given element of Ran p_i. We stated and proved there the necessary and sufficient condition for this problem to have a solution (cf. 4). Here we developed the solution set of this problem.

In 5, we define the concept of reflection $R_{a+p(V)}(x)$ of an element of V with respect to a given affine projection $a + p(x)$, and we prove that for p, q linear commuting projections, the composition of the reflection operators is a reflection and a translation which commute.

Finally, in 6, for p, q arbitrary linear projections in V which may not commute, and P, Q the corresponding affine projections, then $PQP = PQ$ if and only if $pqp = pq$.

(*) Indirizzo: Department of Mathematics, Univ. of Michigan, Ann Arbor, USA-Michigan 48109.

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2 - Notations

Let V be a linear vector space with no topology assigned. As usual we say that p: $V \to V$ is a *linear projection* if p is linear and $pp = p$, and that two maps X, Y: $V \to V$ are *orthogonal* if $XY = YX = 0$. If p, q: $V \to V$ are linear projections that commute under composition, then we see that the four operators pq, $p(I-q)$, $(I-p)q$, $(I-p)(I-q)$ are two by *two orthogonal* and add to I. Then, by the corollary to Theorem 3 of Wilde $[2]_1$, we know that the direct sum of their ranges is V, i.e.

$$V = pq(V) \oplus p(I-q)(V) \oplus (I-p)q(V) \oplus (I-p)(I-q)(V).$$

Thus, if

$$V_{11} = pq(V) \qquad V_{10} = p(I-q)(V) \qquad V_{01} = (I-p)q(V) \qquad V_{00} = (I-p)(I-q)(V)$$

then each x in V is of the form

$$x = x_{11} + x_{10} + x_{01} + x_{00} \qquad \text{where } x_{ij} \text{ is in } V_{ij}.$$

For a in V, then $a + p(x)$ is said to be an *affine projection* if $a + p(a + p(x)) = a + p(x)$. A necessary and sufficient condition for this to occur is that a is in range $(I-p)$, i.e. a is in $(I-p)(V)$.

Note that, if a' is not in $(I-p)(V)$, then $a' + p(V) = (I-p)(a') + p(V)$. In other words, $a' + p(x)$ is not an affine projection, though $(I-p)(a') + p(x)$ is an affine projection: the two operations $a' + p(x)$ and $(I-p)(a') + p(x)$ have the same range but they are different operations.

For a, b in V, $a + p(x)$ and $b + q(x)$ are affine projections if a is in $(I-p)(V)$ and b is in $(I-q)(V)$. Moreover, $a = a_{00} + a_{01}$ and $b = b_{00} + b_{10}$, since $a_{10} = a_{11} = b_{01} = b_{11} = 0$.

3 - Sums of affine subspaces

We want to prove the following

Theorem 1. *If $a + p(x)$ and $b + q(x)$ are affine projections, then*

$$[a + p(V)] + [b + q(V)] = a_{00} + b_{00} + (p + q - pq)(V)$$

where $a_{00} + b_{00} + (p + q - pq)(x)$ *is an affine projection.*

Proof. Let x be in $a + p(V)$ and let y be in $b + q(V)$, hence $x = (a + p)x = a + p(x)$ and $y = (b + q)y = b + q(y)$. Then

$$a_{00} + b_{00} + (p + q - pq)(x + y)$$

$$= a_{00} + b_{00} + (p + q - pq)(a + p(x) + b + q(y)) = a + p(x) + b + q(y) = x + y .$$

Thus

$$(a + p(V)) + (b + q(V)) \subset a_{00} + b_{00} + (p + q - pq)(V) .$$

Now let $x = a_{00} + b_{00} + (p + q - pq)(x)$, let $A(x) = a - b_{10} + p(x)$ and let $B(x) = b - a_{01} + (q - pq)(x)$. Then $A(x) + B(x) = a_{00} + b_{00} + (p + q - pq)(x)$, $A(x)$ is in $a + p(V)$ and $B(x)$ is in $b + q(V)$. So

$$a_{00} + b_{00} + (p + q - pq)(V) \subset (a + p(V)) + (b + q(V)) .$$

Thus the sum of the ranges of two affine projections is the range of an affine projection.

4 - Intersection of affine subspaces

In Wilde $[2]_1$ we solved equations of the form

$$p_i(x) = k_i \qquad (i = 1, \ldots, n) \tag{*}$$

where $p_1, \ldots, p_n$ are n pairwise commuting, linear projections on V and k_i is in $p_i(V)$ for $i = 1, \ldots, n$. We found that a solution exists if and only if $p_i(k_j) = p_j(k_i)$ for $i, j = 1, \ldots, n$.

Now x is in $a + p(V)$ if and only if

$$(I - p)(x) = a \tag{1.1}$$

and x is in $b + q(V)$ if and only if

$$(I - q)(x) = b . \tag{1.2}$$

Thus by Wilde $[2]_1$ $a+p(V)$ and $b+q(V)$ intersect if and only if $(I-p)(b)=(I-q)(a)$. But $(I-p)(b)=b_{00}$ and $(I-q)(a)=a_{00}$. So the subspaces intersect if and only if $b_{00}=a_{00}$. Also, by Wilde $[2]_1$ the intersection is $a+p(b)+pq(V)=a_{00}+a_{01}+b_{10}+pq(V)$ if $b_{00}=a_{00}$. So we have proved the following theorem.

Theorem 2. *If $a+p(x)$ and $b+q(x)$, x in V, are affine projections, then*

$$[a+p(V)]\cap[b+q(V)]=\begin{cases} a_{00}+a_{01}+b_{10}+pq(V) & \text{if } b_{00}=a_{00} \\ \emptyset & \text{if } b_{00}\neq a_{00} \end{cases}$$

where $a_{00}+a_{01}+b_{10}+pq(x)$ is also an affine projection.

In the following expressions and theorems, the operators are supposed to be affine projections. We have the

Corollary. *If p and q are linear projections which commute, then $a+p(V)$ and $b+q(V)$ intersect if and only if $a+p(x)$ and $b+q(x)$ commute under composition, and the intersection is*

$$a+p(b+q(V))=b+q(a+p(V)).$$

Proof. $a+p(b+q(x))=a_{00}+a_{01}+b_{10}+pq(x)$, $b+q(a+p(x))=b_{00}+a_{01}+b_{10}+pq(x)$. Thus $a+p(x)$ and $b+q(x)$ commute if and only if $b_{00}=a_{00}$, i.e., when the subspaces $a+p(V)$ and $b+q(V)$ intersect. The intersection is

$$a_{00}+a_{01}+b_{10}+pq(V)=a+p(b+q(V))=b+q(a+p(V)).$$

Going back to equations (*) in Wilde $[2]_1$ the solution set of $p_i(x)=k_i$ is $k_i+(I-p_i)(V)$. By the corollary, each pair of subspaces $k_i+(I-p_i)(V)$ and $k_j+(I-p_j)(V)$ intersect if and only if $k_i+(I-p_i)(x)$ and $k_j+(I-p_j)(x)$ commute under composition, and the intersection is

$$k_i+(I-p_i)(k_j+(I-p_j)(V))=k_i+(I-p_i)(k_j)+(I-p_i)(I-p_j)(V).$$

Equations (*) are solvable if and only if each pair of these subspaces intersect; and we compose all n projections in any order to get

$$Ik_1 + \sum_{m=2}^{n} [\prod_{j=1}^{m-1} (I-p_j)](k_m)] + (I-p_1)\dots(I-p_n)(V)$$

the common solution of equations (*).

We use the corollary to prove more theorems.

Theorem 3. *$b_{10}+a_{01}+(I-p)(I-q)(V)$ intersect $a+p(V)$ and $b+q(V)$ at $b_{10}+a_{01}+a_{00}$ and $b_{10}+a_{01}+b_{00}$ respectively.*

Theorem 4. *$b_{10}+a_{01}+(I-p-q+2pq)(V)$ intersects $a+p(V)$ and $b+q(V)$ at $b_{10}+a_{01}+a_{00}+pq(V)$ and $b_{10}+a_{01}+b_{00}+pq(V)$ respectively. (The latter two subspaces are parallel.)*

Suppose there exists a linear projection $r\colon V\to V$ such that $a_{00}-b_{00}$ is in $r(V)$ and $pr=rp=qr=rq=0$. Then we can prove the following.

Theorem 5. *$a+p(V)$ and $b+q(V)$ both lie in $(I-r)(a_{00})+(p+q-pq+r)(V)$.*

Theorem 6. *$(I-r)(a_{00})+(p+q-pq+r)(V)$ intersects $b_{10}+a_{01}+(I-p)(I-q)(V)$ and $b_{10}+a_{01}+(I-p-q+2pq)(V)$ at $b_{10}+a_{01}+(I-r)(a_{00})+r(V)$ and $b_{10}+a_{01}+(I-r)(a_{00})+(pq+r)(V)$ respectively.*

Theorem 7. *$b_{10}+a_{01}+(I-r)(a_{00})+r(V)$ intersects $a+p(V)$ and $b+q(V)$ at $b_{10}+a_{01}+a_{00}$ and $b_{10}+a_{01}+b_{00}$ respectively; and $b_{10}+a_{01}+(I-r)(a_{00})+(pq+r)(V)$ intersects $a+p(V)$ and $b+q(V)$ at $b_{10}+a_{01}+a_{00}+pq(V)$ and $b_{10}+a_{01}+b_{00}+pq(V)$ respectively.*

5 - Reflection about subspaces

We define the *reflection $R_{a+p(V)}(x)$ about $a+p(V)$* by taking

$$R_{a+p(V)}(x)-a-p(x) = -(x-a-p(x)) \quad \text{or} \quad R_{a+p(V)}(x) = 2a+(2p-I)(x)$$

an affine involution (cf. Wilde [2]$_2$). We can easily prove the following

Theorem 8. $$R_{b+q(V)}(R_{a+p(V)}(x))$$

$$= R_{b_{10}+a_{01}+(I-p-q+2pq)(V)}(x) + 2b_{00} - 2a_{00} = R_{b_{10}+a_{01}+(I-p-q+2pq)(V)}(x + 2b_{00} - 2a_{00})\,.$$

The result is a reflection and a translation which commute.

6 - Composition property of affine projections

Finally, we prove another theorem. Here p, q denote arbitrary linear projections in V which may not commute.

Theorem 9. *$a+p(b+q(a+p(x))) = a+p(b+q(x))$ for all x in V if and only if $pqp(x) = pq(x)$ for all x in V.*

Proof. The first equation holds for all x in V if and only if $pq(a) + pqp(x) = pq(x)$ for all x in V. If $x = 0$, then $pq(a) = 0$, and $pqp(x) = pq(x)$ for all x in V. If, however, $pqp(x) = pq(x)$ for all x in V, then $pq(a) = pqp(a) = 0$, since a is in $(I-p)(V)$; and so the first equality holds.

If $pq = qp$, then $pqp = ppq = pq$; so if $pq = qp$, then $a + p(b + q(a + p(x))) = a + p(b + q(x))$.

As an example of a vector space V and two linear projections p, q on V such that $pqp = pq$, let $V = \{f/f\colon C^4 \to C\}$, let $p(f)(z_1, z_2, z_3, z_4) = f(z_1, z_2, z_3, z_4) - f(1, 1, z_3, z_4)$, and let $q(f)(z_1, z_2, z_3, z_4) = f(z_1, z_2, z_3, z_4) - f(z_1, 0, 0, z_4)$.

Note that $pqp = pq$ if and only if

$$(I-p)(I-q)(I-p) = (I-q)(I-p)\,.$$

Bibliography

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Sunto

Sia V uno spazio vettoriale complesso senza una topologia assegnata, e siano p e q due proiezioni lineari, commutabili, su V. Se a, $x \in V$, $a + p(x)$ è una proiezione affine su V se e solo se $a \in \mathrm{Im}(I - p)$ e $b + q(x)$ è una proiezione affine su V se e solo se $b \in \mathrm{Im}(I - q)$. In queste ipotesi, il presente lavoro caratterizza la somma di $a + p(V)$ e di $b + q(V)$ come il codominio di un'altra proiezione affine. Inoltre, si danno condizioni necessarie e sufficienti perché $a + p(V)$ e $b + q(V)$ si intersechino; se l'intersezione è non vuota essa è caratterizzata come immagine di una proiezione affine. Altra proprietà di proiezioni affini sono studiate.

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