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A note on temperature, density and velocity in a gas (**)

1 - Introduction

In this Note we establish some results concerning the relations between temperature, number density and macroscopic velocity for a simple monoatomic perfect gas, under suitable conditions; this subject was also studied by one of the authors in his thesis [1]. The main results consists in a simple expression for the scalar field of the (absolute) gas temperature $T(\boldsymbol{r}, t)$ in terms of the scalar field of number density $n(\boldsymbol{r}, t)$ and the vector field of mean velocity $\hat{\boldsymbol{c}}(\boldsymbol{r}, t)$, and an equally simple expression for the number density in terms of temperature and mean velocity, obtained on the basis of both the microscopic approach and the macroscopic one.

Starting with the Boltzmann equation and following the Chapman-Enskog procedure [3], we can obtain continuity, moment and thermal energy equations; on the other hand, we take into account the heat diffusion equation. Combining these two different descriptions of gas dynamics we shall reach our aim.

2 - The Chapman-Enskog procedure

Let $f(\boldsymbol{r}, \boldsymbol{c}, t)$ be the gas velocity distribution function, satisfying the Boltzmann equation

$$\frac{\partial f}{\partial t} + \boldsymbol{c} \cdot \frac{\partial f}{\partial \boldsymbol{r}} + \boldsymbol{F} \cdot \frac{\partial f}{\partial \boldsymbol{c}} = J(f) . \tag{1}$$

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In (1) t, $\boldsymbol{r}$, $\boldsymbol{c}$ are time, position and velocity variables, respectively, $\boldsymbol{F}(\boldsymbol{r}, t)$ in an external force per unit mass, $\frac{\partial}{\partial \boldsymbol{r}}$ and $\frac{\partial}{\partial \boldsymbol{c}}$ are the gradient operators in the position and in the velocity space, respectively, and J is the so called *collision integral.*

Now, suppose $g(\boldsymbol{c})$ is any (scalar, vector or tensor) function of the molecular velocity $\boldsymbol{c}$ (which can depend also on $\boldsymbol{r}$, t and possibly other variables), multiply both sides of (1) by $g\,\mathrm{d}\boldsymbol{c}$ and integrate throughout velocity space; if all the integrals obtained are convergent and the products like gf tend to zero as $\boldsymbol{c}$ tends to infinity in any direction, we obtain

$$(2) \qquad \frac{\partial(n\overline{g})}{\partial t} = -\frac{\partial}{\partial \boldsymbol{r}}\cdot(n\overline{g\boldsymbol{c}}) + n\Big(\overline{\frac{\partial g}{\partial t}} + \overline{\boldsymbol{c}\cdot\frac{\partial g}{\partial \boldsymbol{r}}} + \overline{\boldsymbol{F}\cdot\frac{\partial g}{\partial \boldsymbol{c}}} + \Delta\overline{g}\Big)$$

when $n(\boldsymbol{r}, t)$ is the number density and the «bar» indicates the mean value, i.e., for any Φ,

$$(3) \qquad \overline{\Phi} = \frac{1}{n}\int_{c} \Phi f\,\mathrm{d}\boldsymbol{c}\,.$$

Moreover

$$(4) \qquad \Delta\overline{g} = \frac{1}{n}\int_{c} gJ(f)\,\mathrm{d}\boldsymbol{c}$$

is the mean variation of g per unit time due to collisions.
Introducing the mean (or drift) velocity

$$(5) \qquad \hat{\boldsymbol{c}} = \frac{1}{n}\int_{c} \boldsymbol{c}f\,\mathrm{d}\boldsymbol{c}$$

and the peculiar velocity

$$(6) \qquad \boldsymbol{C} = \boldsymbol{c} - \hat{\boldsymbol{c}}$$

we can rewrite (2) as

$$(7) \qquad \frac{\mathrm{D}(n\overline{g})}{\mathrm{D}t} + n\overline{g}\frac{\partial}{\partial \boldsymbol{r}}\cdot\hat{\boldsymbol{c}} + \frac{\partial}{\partial \boldsymbol{r}}\cdot n\overline{g\boldsymbol{C}}$$

$$-n\Big[\overline{\frac{\mathrm{D}g}{\mathrm{D}t}} + \overline{\boldsymbol{C}\cdot\frac{\partial g}{\partial \boldsymbol{r}}} + \overline{\boldsymbol{F}\cdot\frac{\partial g}{\partial \boldsymbol{C}}} - \frac{\mathrm{D}\hat{\boldsymbol{c}}}{\mathrm{D}t}\cdot\overline{\frac{\partial g}{\partial \boldsymbol{C}}} - \overline{\frac{\partial g}{\partial \boldsymbol{C}}\boldsymbol{C}}:\frac{\partial \hat{\boldsymbol{c}}}{\partial \boldsymbol{r}}\Big] = n\Delta\overline{g}$$

where

(8) $$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \hat{\boldsymbol{c}} \cdot \frac{\partial}{\partial \boldsymbol{r}}$$

is the *mobile operator* (or *substantial time derivative*); here and in the sequel we denote $\boldsymbol{AB}$ for the tensor product of the vectors $\boldsymbol{A}$ and $\boldsymbol{B}$, and denote $\mathbf{A}:\mathbf{B}$ the double scalar product between the tensors $\mathbf{A}$ and $\mathbf{B}$ (i.e. $\mathbf{A}:\mathbf{B} = \sum_{i=1}^{3} \sum_{j=1}^{3} A_{ij} B_{ji}$). We can name equation (2) or (7) the Chapman-Enskog equation. If we take g equal to each fundamental collision invariant, $\Delta \overline{g}$ vanishes and we obtain, after some calculation, the following:

(a) when $g \equiv 1$, the continuity equation

(9) $$\frac{\partial n}{\partial t} + \frac{\partial}{\partial \boldsymbol{r}} \cdot (n\hat{\boldsymbol{c}}) = 0\,;$$

(b) when $g \equiv \boldsymbol{C}$, the Euler (or moment) equation

(10) $$\frac{\partial}{\partial \boldsymbol{r}} \cdot \mathbf{p} = \rho\left(\boldsymbol{F} - \frac{\mathrm{D}\hat{\boldsymbol{c}}}{\mathrm{D}t}\right)$$

where $\rho = nm$ is the mass density and $\mathbf{p}$ is the kinetic pressure tensor, defined by

(11) $$\mathbf{p} = \rho\overline{\boldsymbol{CC}}\,;$$

(c) when $g = \frac{1}{2}mC^2$, the thermal energy equation

(12) $$\frac{\mathrm{D}T}{\mathrm{D}t} = -\frac{2}{3kn}\left(\mathbf{p} : \frac{\partial \hat{\boldsymbol{c}}}{\partial \boldsymbol{r}} + \frac{\partial}{\partial \boldsymbol{r}} \cdot \boldsymbol{q}\right)$$

where $\boldsymbol{q}$ is the thermal flow vector, defined by

(13) $$\boldsymbol{q} = n\overline{\frac{1}{2}mC^2\boldsymbol{C}}$$

and T is the absolute temperature, satisfying the equation

(14) $$\overline{\frac{1}{2}mC^2} = \frac{3}{2}kT$$

with k the gas Boltzmann constant.

Now, back to Boltzmann equation (1) following the Chapman-Enskog expansion for its solution [2] [3], we see that, when the validity of the second approximation to f is assumed (i.e., when the gas density is not very large), the following equations hold

$$\mathbf{p} = nkT\mathbf{I}_2 - 2\mu\left(\frac{\partial \hat{\boldsymbol{c}}}{\partial \boldsymbol{r}}\right)^0 \tag{15}$$

$$\boldsymbol{q} = -\lambda \frac{\partial T}{\partial \boldsymbol{r}}. \tag{16}$$

Here, $\mathbf{I}_2$ is the unit dyadic, λ and μ are positive coefficients, which can depend on $\boldsymbol{r}$ and t; moreover, here and after we mean that, for each 2-tensor $\mathbf{A}$,

$$\mathbf{A}^0 = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T) - \frac{1}{3}\left[\frac{1}{2}(\mathbf{A} + \mathbf{A}^T) \colon \mathbf{I}_2\right]\mathbf{I}_2 \qquad A_{ij}^T = A_{ji} \text{ for each } i,\ j) \tag{17}$$

so that $\mathbf{A}^0$ is the *traceless symmetric part* of $\mathbf{A}$. Note that

$$\mathbf{A}^0 \colon \mathbf{A} = \mathbf{A} \colon \mathbf{A}^0 = \mathbf{A}^0 \colon \mathbf{A}^0 . \tag{18}$$

One easily sees that λ and μ can be identified, respectively, with the thermal conductivity and the viscosity coefficients.

3 - Relations between temperature, density and mean velocity

In the previous section, we recalled some results that one can derive by means of the Chapman-Enskog (macroscopic) treatment of the (microscopic) Boltzmann equation. On the other hand, if there is no heat source for our gas, we can write for it, on the basis of a completely macroscopic approach, the heat conduction equation

$$\frac{\partial T}{\partial t} = \frac{1}{sp} \frac{\partial}{\partial \boldsymbol{r}} \cdot \left(\lambda \frac{\partial T}{\partial \boldsymbol{r}}\right) \text{ (1)} \tag{19}$$

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(1) If λ is independent of $\boldsymbol{r}$, i.e. if the gas is thermally homogeneous, eq. (18) is obviously given the more familiar form $\frac{\partial T}{\partial t} = \frac{\lambda}{sp} \nabla^2 T$, where $\nabla^2 = \frac{\partial}{\partial \boldsymbol{r}} \cdot \frac{\partial}{\partial \boldsymbol{r}}$ is the Laplace operator. The same remarks can be made for eqs. (18)′ and (19). It can easily be seen, however, that the final results hold in both cases.

where s is the specific heat, which is given, for a monoatomic gas, by [3]

$$s = \frac{3k}{m} \tag{20}$$

so that eq. (18) can be rewritten as

$$\frac{\partial T}{\partial t} = \frac{2}{3kn} \frac{\partial}{\partial \boldsymbol{r}} \cdot \left(\lambda \frac{\partial T}{\partial \boldsymbol{r}}\right). \tag{19$'$}$$

Now, taking account of eqs. (15) and (16), we obtain from eq. (12)

$$\frac{\partial T}{\partial t} + \hat{\boldsymbol{c}} \cdot \frac{\partial T}{\partial \boldsymbol{r}} = -\frac{2}{3} T \frac{\partial}{\partial \boldsymbol{r}} \cdot \hat{\boldsymbol{c}} + \frac{4\mu}{3kn} \left(\frac{\partial \hat{\boldsymbol{c}}}{\partial \boldsymbol{r}}\right)^0 : \left(\frac{\partial \hat{\boldsymbol{c}}}{\partial \boldsymbol{r}}\right)^0 + \frac{2}{3kn} \frac{\partial}{\partial \boldsymbol{r}} \cdot \left(\lambda \frac{\partial T}{\partial \boldsymbol{r}}\right) \tag{21}$$

so that we have, by (19)$'$,

$$\hat{\boldsymbol{c}} \cdot \frac{\partial T}{\partial \boldsymbol{r}} = -\frac{2}{3} T \frac{\partial}{\partial \boldsymbol{r}} \cdot \hat{\boldsymbol{c}} + \frac{4\mu}{3kn} \left(\frac{\partial \hat{\boldsymbol{c}}}{\partial \boldsymbol{r}}\right)^0 : \left(\frac{\partial \hat{\boldsymbol{c}}}{\partial \boldsymbol{r}}\right)^0. \tag{22}$$

We can see that, if both $\hat{\boldsymbol{c}}$ and T are independent of $\boldsymbol{r}$, then eq. (22) vanishes, and what follows does not apply. Otherwise we see that, by eqs. (10) and (15), eq. (22) yields finally, setting for simplicity $\mathbf{e} = \left(\frac{\partial \hat{\boldsymbol{c}}}{\partial \boldsymbol{r}}\right)^0$,

$$T = \frac{\frac{4\mu}{3kn} \mathbf{e} : \mathbf{e} - \frac{2\mu}{kn} \hat{\boldsymbol{c}} \cdot \left(\frac{\partial}{\partial \boldsymbol{r}} \cdot \mathbf{e}\right) - \frac{m}{k} \hat{\boldsymbol{c}} \cdot \left(\boldsymbol{F} - \frac{\mathrm{D}\hat{\boldsymbol{c}}}{\mathrm{D}t}\right)}{\frac{2}{3} \frac{\partial}{\partial \boldsymbol{r}} \cdot \hat{\boldsymbol{c}} - \hat{\boldsymbol{c}} \cdot \frac{\partial \log n}{\partial \boldsymbol{r}}} \tag{23}$$

provided that the denominator of the r.h.s. does not vanish. If it vanishes (e.g., when the gas is homogeneous and the mean velocity field is solenoidal), we obtain a constraint on velocity and number density which makes the numerator zero.

Moreover, from (22) we have immediately

$$n = \frac{\frac{4\mu}{3k} \mathbf{e} : \mathbf{e}}{\hat{\boldsymbol{c}} \cdot \frac{\partial T}{\partial \boldsymbol{r}} + \frac{2}{3} T \frac{\partial}{\partial \boldsymbol{r}} \cdot \hat{\boldsymbol{c}}}. \tag{24}$$

Since the denominator of the r.h.s. of (24) can be written as

$$\frac{2}{3} T^{-1/2} \frac{\partial}{\partial \boldsymbol{r}} \cdot (T^{3/2} \hat{\boldsymbol{c}})$$

we have that if $T^{3/2} \hat{\boldsymbol{c}}$ is solenoidal, then $\mathbf{e} : \mathbf{e} \equiv 0$ and viceversa, in which case, as one can see, the tensor $\frac{\partial \hat{\boldsymbol{c}}}{\partial \boldsymbol{r}}$ must be of the form $\alpha \mathbf{I}_2$.

So, two simple relations between temperature, number density and mean (macroscopic) velocity for a (simple) gas are been established under rather general conditions.

References

[1] G. Beltrami, *Temperatura e velocità in un gas*, Tesi di Laurea, Università di Parma, 1986.

[2] C. Cercignani, *The Boltzmann equation and its applications*, Springer, 1988.

[3] S. Chapman and T. G. Cowling, *The mathematical theory of non-uniform gases*, Cambridge, 1970.

Sunto

Vengono stabilite relazioni fra temperatura, densità numerica e velocità macroscopica per un gas semplice, sotto condizioni generali. I risultati sono ottenuti sulla base di considerazioni sia di tipo microscopico che di tipo macroscopico.

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