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Affine connections on manifolds with almost contact 3-structure (**)

Introduction

An almost contact 3-structure $(\varphi_i, \ \xi_i, \ \eta^i)$ $(i=1, \ 2, \ 3)$ on a differentiable manifold M, is an aggregate consisting on three almost contact structures, satisfying certain compatibility conditions (v. **1**).

In this paper, we shall study affine connections on manifolds with almost contact structures and with almost contact 3-structures.

Among other results, we obtain a necessary and sufficient condition for the existence of a symmetric $(\varphi_i, \ \xi_i, \ \eta^i)$-connection in terms of the tensor fields φ_i, of the 1-forms η^i and of the Nijenhuis tensor fields N_{φ_i}.

1 - Preliminaries

An almost contact structure $(\varphi, \ \xi, \ \eta)$ [5] on a differentiable manifold is an aggregate consisting on a tensor field φ of type (1, 1), a vector field ξ, and a 1-form η which satisfy

$$\varphi^2 = -I + \eta \otimes \xi \qquad \eta(\xi) = 1$$

where $\otimes$ denotes the tensor product, and I is the identity tensor.

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Let us suppose that a differentiable manifold admits three almost contact structures $(\varphi_i, \xi_i, \eta^i)$ $(i=1, 2, 3)$ satisfying

$$\eta^i(\xi_j)=\delta^j_i \qquad \varphi_i(\xi_j)=-\varphi_j(\xi_i)=\xi_k$$

$$\eta^i\circ\varphi_j=-\eta^j\circ\varphi_i=\eta^k \qquad \varphi_i\varphi_j-\eta^j\otimes\xi_i=-\varphi_j\varphi_i+\eta^i\otimes\xi_j=\varphi_k$$

for any cyclic permutation (i, j, k) of $(1, 2, 3)$. Then $(\varphi_i, \xi_i, \eta^i)$ $(i=1, 2, 3)$ is called an almost contact 3-structures [**1**]. The dimension of a manifold with an almost contact 3-structure is $4n+3$ for some non negative integer n.

Let M be a differentiable manifold with an almost contact 3-structure. Then, the structure group of the tangent bundle is reducible to $\mathrm{Sp}(n)\times I_3$, where $\mathrm{Sp}(n)$ denotes the real representation of the symplectic group. The converse is also true.

2 - Affine connections on almost contact manifolds

Let M be a differentiable manifold with an almost contact structure (φ, ξ, η). Two linear operators F_1, F_2 acting on the space of tensor of type (1, 2) are defined as follows.

Def. 2.1. Let P be a tensor field of type (1, 2).
Then

$$F_1P(X, Y)=\tfrac{1}{2}\{-\varphi^2P(X, Y)-\varphi P(X, \varphi Y)\}$$

$$F_2P(X, Y)=\tfrac{1}{2}\{-\varphi^2P(X, Y)+\varphi P(X, \varphi Y)\}$$

for any vector fields X, Y on M.

Proposition 2.1. *The following identities and implications hold*

$$F_1+F_2=-\varphi^2 \tag{2.1}$$

(2.2)
$$F_1F_1P(X, Y)=F_1P(X, Y)-\tfrac{1}{2}F_1P(X, \eta(Y)\xi)$$
$$F_2F_2P(X, Y)=F_2P(X, Y)-\tfrac{1}{2}F_2P(X, \eta(Y)\xi)$$

(2.3) $$F_1F_2P(X, Y) = F_2F_1P(X, Y) - \frac{1}{2}F_1P(X, \eta(Y)\xi)$$

(2.4) *If* $F_1P(X, Y) = -\frac{1}{2}\varphi^2P(X, \eta(Y)\xi)$ *then*

$$F_2P(X, Y) = -\varphi^2P(X, Y) + \frac{1}{2}\varphi^2P(X, \eta(Y)\xi).$$

Similarly, if $F_2P(X, Y) = -\frac{1}{2}\varphi^2P(X, \eta(Y)\xi)$ *then*

$$F_1P(X, Y) = -\varphi^2P(X, Y) + \frac{1}{2}\varphi^2P(X, \eta(Y)\xi).$$

(2.5) *Given a tensor field P of tupe* (1, 2), *there exists a tensor field Q of type* (1, 2) *such that* $F_1Q(X, Y) = -\varphi^2P(X, Y)$ *if and only if*

$$F_2P(X, Y) = -\frac{1}{2}\varphi^2P(X, \eta(Y)\xi).$$

Similarly, there exists a tensor field Q such that $F_2Q(X, Y) = -\varphi^2P(X, Y)$ *if and only if*

$$F_1P(X, Y) = -\frac{1}{2}\varphi^2P(X, \eta(Y)\xi).$$

Proof. (2.1) and (2.4) follow directly from Def. 2.1; (2.2) and (2.3) are shown by direct computation. Finally, if there exists a tensor field Q such that $F_1Q(X, Y) = -\varphi^2P(X, Y)$, then

$$F_2F_1Q(X, Y)$$

$$= \frac{1}{2}(\varphi^4P(X, Y) - \varphi^3P(X, \varphi Y)) = \frac{1}{2}(-\varphi^2P(X, Y) + \varphi P(X, \varphi Y)) = F_2P(X, Y).$$

Then, by (2.3),

$$F_2F_1Q(X, Y) = \frac{1}{2}F_1Q(X, \eta(Y)\xi) = -\frac{1}{2}\varphi^2P(X, \eta(Y)\xi) = F_2P(X, Y).$$

Conversely, if

$$F_2P(X, Y) = -\frac{1}{2}\varphi^2P(X, \eta(Y)\xi)$$

then, by (2.4),

$$F_1P(X, Y) = -\varphi^2P(X, Y) + \frac{1}{2}\varphi^2P(X, \eta(Y)\xi)$$

and hence

$$F_1(P(X, Y)+P(X, \eta(Y)\xi))=-\varphi^2 P(X, Y).$$

Now, it suffices to take

$$Q(X, Y)=P(X, Y)+P(X, \eta(Y)\xi).$$

Theorem 2.1. *On a differentiable manifold M with an almost contact structure (φ, ξ, η), there always exists an affine connection such that φ, ξ and η, are all covariantly constant with respect to it.*

Proof. Let us consider an arbitrary affine connection $\mathring{\nabla}$ on M. Then, define the connection ∇' by

$$\nabla'_X Y=\mathring{\nabla}_X Y-\eta(Y)\mathring{\nabla}_X \xi$$

for any differentiable vector fields X, Y over M. Then $\nabla'_X \xi=0$.

Next, let us define $\overline{\nabla}$ by

$$\overline{\nabla}_X Y=\nabla'_X Y+((\nabla'_X \eta)Y)\xi$$

Then, $\overline{\nabla}_X \xi=0$ and $(\overline{\nabla}_X \eta)Y=0$.

Finally, define

$$\nabla_X Y=\overline{\nabla}_X Y-\tfrac{1}{2}\varphi(\overline{\nabla}_X \varphi)Y.$$

This connection ∇ satisfies the required conditions.

We call such an affine connection a (φ, ξ, η)-connection. The following proposition is easily obtained.

Proposition 2.2. *Let $\mathring{\nabla}$ be an affine connection on M such that $\mathring{\nabla}\xi=\mathring{\nabla}\eta=0$. Then, an affine connection ∇ on M such that $\nabla\xi=\nabla\eta=0$, is always of the form $\nabla=\mathring{\nabla}+Q$ where Q is a tensor field of type $(1, 2)$ verifying*

(1) $Q(X, \xi)=0$ (2) $\eta(Q(X, Y))=0$

for any vector fields X, Y on M.

Now, let ∇ be an affine connection such that $\nabla\xi=\nabla\eta=0$. If $\nabla\varphi=0$, then

$$0=\varphi(\mathring{\nabla}_X\varphi)Y+\varphi(Q(X,\ \varphi Y))-\varphi^2Q(X,\ Y) \tag{2.6}$$

being Q a tensor field of type (1.2) in the conditions of Proposition 2.1. If we put

$$P(X,\ Y)=-\tfrac{1}{2}\varphi(\mathring{\nabla}_X\varphi)Y$$

then, (2.6) is written as follows

$$P(X,\ Y)=F_2Q(X,\ Y); \tag{2.7}$$

but, $P(X,\ Y)=-\varphi^2P(X,\ Y)$, and then, by (2.5),

$$F_1P(X,\ Y)=-\tfrac{1}{2}\varphi^2P(X,\ \eta(Y)\xi)=0$$

and, by (2.4),

$$F_2P(X,\ Y)=-\varphi^2P(X,\ Y)=P(X,\ Y).$$

Hence, (2.7) is written as follows

$$F_2(Q(X,\ Y)-P(X,\ Y))=0=-\tfrac{1}{2}\varphi^2(Q(X,\ \eta(Y)\xi)-P(X,\ \eta(Y)\xi)).$$

Then, from (2.5), there exists a tensor field A such that

$$F_1A(X,\ Y)=-\varphi^2(Q(X,\ Y)-P(X,\ Y))=Q(X,\ Y)-P(X,\ Y).$$

Therefore Q must be of the form

$$Q(X,\ Y)=P(X,\ Y)+F_1(A(X,\ Y))$$

and $Q(X,\ \xi)=0$, i.e. $\varphi A(X,\ \xi)=0$.

Conversely, if $\mathring{\nabla}$ is an affine connection on M such that $\mathring{\nabla}\xi=\mathring{\nabla}\eta=0$, and if A is a tensor field of type (1, 2) satisfying $\varphi A(X,\ \xi)=0$, then the connection ∇ defined by

$$\nabla_XY=\mathring{\nabla}_XY-\tfrac{1}{2}\varphi(\mathring{\nabla}_X\varphi)Y+F_1A(X,\ Y)$$

is a (φ, ξ, η)-connection. Thus, we have

Theorem 2.2. *Let* $\mathring{\nabla}$ *be an affine connection on* M *such that* $\mathring{\nabla}\xi = \mathring{\nabla}\eta = 0$. *Then, every* (φ, ξ, η)*-connection on* M *is of the form*

$$\nabla_X Y = \mathring{\nabla}_X Y - \tfrac{1}{2}\varphi(\mathring{\nabla}_X \varphi) Y + F_1 A(X, Y)$$

being A *a tensor field of type* $(1, 2)$ *such that* $\varphi A(X, \xi) = 0$ *for any vector fields* X, Y *on* M.

Corollary 2.1. *Let* ∇ *be a* (φ, ξ, η)*-connection on* M. *Then, a connection* $\widetilde{\nabla}$ *of the form*

$$\widetilde{\nabla}_X Y = \nabla_X Y + H(X, Y)$$

being H *a tensor field of type* $(1, 2)$, *is a* (φ, ξ, η)*-connection, if and only if* $H(X, \xi) = 0$ *and* $H(X, Y) = F_1 B(X, Y)$ *for some tensor field* B *of type* $(1, 2)$.

Proof. Let $\mathring{\nabla}$ be a (φ, ξ, η)-connection. Then,

$$\widetilde{\nabla}_X Y = \mathring{\nabla}_X Y - \tfrac{1}{2}\varphi(\mathring{\nabla}_X \varphi) Y + F_1 A(X, Y)$$

with $\varphi A(X, \xi) = 0$. On the other hand

$$\nabla_X Y + H(X, Y) = \mathring{\nabla}_X Y - \tfrac{1}{2}\varphi(\mathring{\nabla}_X \varphi) Y + F_1 A'(X, Y) + H(X, Y)$$

with $F_1 A'(X, \xi) = 0$. Therefore,

$$H(X, Y) = F_1(A(X, Y) - A'(X, Y)) = F_1 B(X, Y)$$

being $B = A - A'$ and $H(X, \xi) = F_1(A(X, \xi) - A'(X, \xi)) = 0$. Now, if

$$\widetilde{\nabla}_X Y = \nabla_X Y + H(X, Y)$$

with $H(X, \xi) = 0$ and $H(X, Y) = F_1 B(X, Y)$, then

$$\widetilde{\nabla}_X Y = \mathring{\nabla}_X Y - \tfrac{1}{2}\varphi(\mathring{\nabla}_X \varphi) Y + F_1(A(X, Y) + B(X, Y))$$

$$F_1(A(X, \xi) + B(X, \xi)) = 0$$

hence $\widetilde{\nabla}$ is a (φ, ξ, η)-connection.

Theorem 2.3. *Let* (φ, ξ, η) *be an almost contact structure on* M. *Then, there exists a symmetric* (φ, ξ, η)*-connection if and only if* $N_\varphi = 0$ *and* $d\eta = 0$, *being* N_φ *the Nijenhuis tensor of* φ

$$N_\varphi(X, Y) = -\varphi^2[X, Y] + \varphi[\varphi X, Y] + \varphi[X, \varphi Y] - [\varphi X, \varphi Y].$$

Proof. If ∇ is a symmetric (φ, ξ, η)-connection, then it follows immediately that $N\varphi = 0$ and $d\eta = 0$.

Conversely, assume that $N\varphi = 0$ and $d\eta = 0$. Then we define a symmetric (φ, ξ, η)-connection as follows.

Let $\mathring{\nabla}$ be an affine connection such that $\mathring{\nabla}\xi = \mathring{\nabla}\eta = 0$, and such that its torsion tensor is given [3] by $T(X, Y) = 2d\eta \otimes \xi$.

Then, ∇ given by

$$\nabla_X Y = \mathring{\nabla}_X Y - \tfrac{1}{2}\varphi(\mathring{\nabla}_X \varphi) Y$$

is a (φ, ξ, η)-connection, and the torsion tensor S of ∇ is given by

$$S(X, Y) = T(X, Y) - \tfrac{1}{2}(\varphi(\mathring{\nabla}_X \varphi) Y - \varphi(\mathring{\nabla}_Y \varphi) X).$$

But $d\eta = 0$ implies $T = 0$, and therefore

$$S(X, \xi) = -\tfrac{1}{2}(\varphi(\mathring{\nabla}_X \varphi)\xi - \varphi(\mathring{\nabla}_\xi \varphi) X) = \tfrac{1}{2}\varphi(\mathring{\nabla}_\xi \varphi X - \varphi\mathring{\nabla}_\xi X)$$

$$= \tfrac{1}{2}\varphi(-[\varphi X, \xi] + \varphi[X, \xi]) = \tfrac{1}{2}(\varphi^2[X, \xi] - \varphi[\varphi X, \xi]) = -\tfrac{1}{2}N\varphi(X, \xi) = 0.$$

Hence, by Corollary (2.1), the connection $\overline{\nabla}$ given by

$$\overline{\nabla}_X Y = \nabla_X Y - F_1 F_4 S(X, Y)$$

is a (φ, ξ, η)-connection, being F_4 the linear operator over the space of tensors of type (1, 2) defined by

$$F_4 P(X, Y) = \tfrac{1}{2}\{-\varphi^2 P(X, Y) + P(\varphi X, \varphi Y)\}.$$

Denoting by $\overline{T}$ the torsion tensor of $\overline{\nabla}$ we obtain

$$\overline{T}(X, Y) = S(X, Y) - F_1 F_4 S(X, Y) + F_1 F_4 S(Y, X)$$

$$= S(X, Y) - \tfrac{1}{4}\{\varphi^4 S(X, Y) + \varphi^3 S(X, \varphi Y) - \varphi^2 S(\varphi X, \varphi Y)$$

$$- \varphi S(\varphi X, \varphi^2 Y)\} + \tfrac{1}{4}\{\varphi^4 S(Y, X) + \varphi^3 S(Y, \varphi X) - \varphi^2 S(\varphi Y, \varphi X) - \varphi S(\varphi Y, \varphi^2 X)\}$$

$$= S(X, Y) - \tfrac{1}{4}\{S(X, Y) + \varphi S(X, \varphi Y) + S(\varphi X, \varphi Y)$$

$$+ \varphi S(\varphi X, Y)\} + \tfrac{1}{4}\{S(Y, X) + \varphi S(Y, \varphi X) + S(\varphi Y, \varphi X) + \varphi S(\varphi Y, X)\}$$

$$= \tfrac{1}{2} S(X, Y) - \tfrac{1}{2} S(\varphi X, \varphi Y).$$

If now we define

$$F_3 P(X, Y) = \tfrac{1}{2}\{-\varphi^2 P(X, Y) - P(\varphi X, \varphi Y)\}$$

then

$$\overline{T}(X, Y) = F_3 S(X, Y).$$

But

$$F_3 S(X, Y) = \tfrac{1}{2} F_3(\varphi(\mathring{\nabla}_X \varphi) Y - \varphi(\mathring{\nabla}_Y \varphi) X)$$

$$= -\tfrac{1}{4}\{-\varphi^3(\mathring{\nabla}_X \varphi) Y - \varphi(\mathring{\nabla}_{\varphi X} \varphi) \varphi Y - (-\varphi^3(\mathring{\nabla}_Y \varphi) X - \varphi(\mathring{\nabla}_{\varphi Y} \varphi) X)\}$$

$$= -\tfrac{1}{4}\{\varphi(\mathring{\nabla}_X \varphi) Y - (\mathring{\nabla}_{\varphi X} \varphi) \varphi Y - \varphi(\mathring{\nabla}_Y \varphi) X + (\mathring{\nabla}_{\varphi Y} \varphi) X\}$$

$$= -\tfrac{1}{4} N\varphi(X, Y) = 0$$

and the proof is complete.

3 - Affine connections on manifolds with an almost contact 3-structure

Lemma 3.1. *On a differentiable manifold M, with an almost contact 3-structure* $(\varphi_i, \xi_i, \eta^i)$ $(i = 1, 2, 3)$ *there always exists an affine connection such that* φ_i, ξ_i, η^i *are covariantly constant with respect to it.*

We call such an affine connection a $(\varphi_i, \xi_i, \eta^i)$-connection.

Proof. Given a lineal connection $\overset{\circ}{\nabla}$, define

$$\nabla'_X Y = \overset{\circ}{\nabla}_X Y - \sum_{i=1}^{3} \eta^i(Y) \overset{\circ}{\nabla}_X \xi_i .$$

Then, $\nabla'_X \xi_i = 0$ $(i=1, 2, 3)$. Now

$$\overline{\nabla}_X Y = \nabla'_X Y + \sum_{i=1}^{3} ((\nabla'_X \eta^i) Y) \xi_i$$

satisfies $\overline{\nabla}_X \xi = \overline{\nabla}_X \eta^i = 0$ $(i=1, 2, 3)$. Finally

$$\nabla_X Y = \overline{\nabla}_X Y - \tfrac{1}{4} \sum_{i=1}^{3} \varphi_i(\overline{\nabla}_X \varphi_i) Y$$

is a $(\varphi_i, \xi_i, \eta^i)$-connection.

By using the almost contact 3-structure $(\varphi_i, \xi_i, \eta^i)$ $(i=1, 2, 3)$ we define linear operators $F_1, F_2, F_3, F_4, G_1, G_2, G_3, G_4, H_1, H_2, H_3, H_4$, respectively, in the same way as we did in 2.

The following proposition is easily proved.

Proposition 3.1. *Let* ∇ *be a* $(\varphi_i, \xi_i, \eta^i)$*-connection. Let be*

$$\hat{\nabla}_X Y = \nabla_X Y + H(X, Y)$$

being H *a tensor field of type* $(1, 2)$. *Then* $\hat{\nabla}$ *is a* $(\varphi_i, \xi_i, \eta^i)$*-connection if and only if*

$$F_2 H(X, Y) = G_2 H(X, Y) \qquad H(X, \xi_i) = 0 \qquad \eta^i(H(X, Y)) = 0 \quad (i=1, 2, 3)$$

for any vector fields X, Y *on* M.

Proposition 3.2. *Let* H *be a tensor field of type* $(1, 2)$. *Then* $\eta^i(H(X, Y)) = 0$, $H(X, \xi_i) = 0$ *and* $F_2 H = G_2 H = 0$ *if and only if there exists a tensor field* Q *of type* $(1, 2)$ *such that*

$$H(X, Y) = F_1 G_1 Q(X, Y) \qquad Q(X, \xi_i) = 0 \qquad \eta^i(Q(X, Y)) = 0 .$$

Proof. If $F_2H(X, Y)=0$ and $H(X, \xi_i)=0$, then, by (2.4), $F_1H(X, Y)=H(X, Y)$.

If $G_2H(X, Y)=0$, then, $G_1H(X, Y)=H(X, Y)$. Therefore $F_1G_1H(X, Y)=F_1H(X, Y)=H(X, Y)$ and it is sufficient to take $Q=H$.

Conversely, if there exists a tensor field Q satisfying $H(X, Y)=F_1G_1Q(X, Y)$, $Q(X, \xi_i)=0$ and $\eta^i(Q(X, Y))=0$, then, $H(X, \xi_i)=0$ and $\eta^i(H(X, Y))=0$.

Finally, from (2.5), $H(X, Y)=F_1G_1Q(X, Y)=G_1F_1Q(X, Y)$ imply $F_2H(X, Y)=0$, $G_2H(X, Y)=0$.

Using Propositions 3.1 and 3.2 we can state

Theorem 3.1. *Let ∇ be a $(\varphi_i, \xi_i, \eta^i)$-connection. Then*

$$\widetilde{\nabla}_X Y=\nabla_X Y+H(X, Y)$$

is a $(\varphi_i, \xi_i, \eta^i)$-connection if and only if there exists a tensor field Q of type (1, 2) *such that*

$$Q(X, \xi_i)=0 \qquad \eta^i(Q(X, Y)=0 \qquad H(X, Y)=F_1G_1Q(X, Y)$$

for any vector fields X, Y on M.

The next proposition is easily obtained.

Proposition 3.3. *Let P be a tensor field of type* (1, 2) *such that $P(X, \xi)=\eta^i(P(X, Y))=0$. Then*

$$F_1G_1P(X, Y)=G_1H_1P(X, Y)=H_1F_1P(X, Y)$$

$$=G_1F_1P(X, Y)=H_1G_1P(X, Y)=F_1H_1P(X, Y)$$

$$=\tfrac{1}{4}\{P(X, Y)-\varphi_1P(X, \varphi_1Y)-\varphi_2P(X, \varphi_2Y)-\varphi_3P(X, \varphi_3Y)\}$$

hold good for any vector fields X, Y on M.

Theorem 3.2. *Let $(\varphi_i, \xi_i, \eta^i)$ $(i=1, 2, 3)$ be an almost contact 3-structure on M. Then, there exists a symmetric $(\varphi_i, \xi_i, \eta^i)$-connection if and only if $N_{\varphi_i}=0$, $d\eta^i=0$ and $\mathfrak{L}_{\xi_j}\varphi_i=0$, where $\mathfrak{L}$ denotes the Lie differentiation.*

Proof. If ∇ is a symmetric $(\varphi_i, \xi_i, \eta^i)$-connection, then the conditions $N_{\varphi_i}=0$, $d\eta^i=0$, and $\mathfrak{L}_{\xi_j}\varphi_i=0$ follow immediately. Now, suppose

$$N_{\varphi_i}=0 \qquad d\eta^i=0 \qquad \mathfrak{L}_{\xi_j}\varphi_i=0.$$

Let $\mathring{\nabla}$ be an affine connection such that $\mathring{\nabla}\xi_i=0$, $\mathring{\nabla}\eta^i=0$, and with torsion tensor [6]

$$\mathring{T}=\sum_{i=1}^{3} d\eta^i\otimes\xi_i.$$

Then $d\eta^i=0$ implies $\mathring{T}=0$. Define

$$\nabla_X Y=\mathring{\nabla}_X Y-\tfrac{1}{4}\sum_{i=1}^{3}\varphi_i(\mathring{\nabla}_X\varphi_i)Y.$$

Then, ∇ is a $(\varphi_i, \xi_i, \eta^i)$-connection and its torsion tensor is given by

$$S(X, Y)=-\tfrac{1}{4}\{\sum_{i=1}^{3}\varphi_i(\mathring{\nabla}_X\varphi_i)Y-\sum_{i=1}^{3}\varphi_i(\mathring{\nabla}_Y\varphi_i)X\}.$$

A straightforward computation shows that $S(X, \xi_i)=0$ and $\eta^i S(X, Y)=0$; therefore, from Theorem 2.1, if

$$\nabla'_X Y=\nabla_X Y-4G_1F_1F_4S(X, Y)$$

then ∇' is a $(\varphi_i, \xi_i, \eta^i)$-connection. But the torsion tensor of ∇' is given by $S'(X, Y)=F_3(N\varphi_2+N\varphi_3)$ and, therefore, ∇' is symmetric.

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Summary

See Introduction.

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