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Semigroup identities in near-rings (**)

Introduction

This paper is a continuation of the researches of [2] on various kinds of identities in (left) near-rings.

We have seen in $[2]_1$ that an autodistributive and idempotent near-ring satisfies the identity $xyz = yxz$. Here we show (Theorem 2.6) that in a large class of near-rings (including the regular ones) any permutation identity implies the $xyz = yxz$.

1 - Preliminaries

In this paper we work on *left* near-rings. For such a N we put $A(x) = \{y \in N | xy = 0\}$ for each $x \in N$, and $A(X) = \cap \{A(x) | x \in X\}$ for each $X \subseteq N$. Moreover, we put $K = \{x \in N | A(x) \neq 0\}$ and $A = \cap \{A(x) | x \in K\}$.

A near-ring N is called *simple* if it has no non-trivial ideals, *integral* if it is 0-symmetric and without divisors of zero, *regular* if for each $x \in N$ there is $x' \in N$ such that $xx'x = x$, *neutral* if for each $x, y \in N$, $A(x) \neq 0$ implies $xy = 0y$. It is said that N has the IFP if $xy = 0$ implies $xry = 0$ for each $x, y, r \in N$.

We call here N a *W-near-ring* if there is a map $x \mapsto x^0$ from N to itself, such that $(x^0)^2 = x^0$ and $x^0 x = x$ for all x and, moreover, for each N-subgroup M of N, $x \in M$ implies $x^0 \in M$. This concept is similar but different from that of weak regularity.

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A permutation identity $x_1 x_2 \dots x_n = x_{\varphi(1)} x_{\varphi(2)} \dots x_{\varphi(n)}$ will be denoted by $I(\varphi)$ (n is supposed to be fixed). These identities have been studied, e.g., in [1] and $[3]_2$.

The following has been proved (see $[3]_1$, Thm. 1 or [1], Lemma 2).

Lemma 1.1. *Let S be a semigroup satisfying $I(\varphi)$. For each $a_1, \dots, a_n$, $b_1, \dots, b_n$, x, $y \in S$ we get*

$$a_1 \dots a_n xy b_1 \dots b_n = a_1 \dots a_n yx b_1 \dots b_n .$$

Lemma 1.2. *Let N be a near-ring, let k be a nonnegative integer, let N' be the set of all products of k elements of N. If N satisfies $I(\varphi)$, where $\varphi(i) = i$ for $i \leqslant k$, then $N/A(N')$ satisfies the identity*

$$x_{k+1} x_{k+2} \dots x_n = x_{\varphi(k+1)} x_{\varphi(k+2)} \dots x_{\varphi(n)} .$$

Proof. Let $b \in N'$ and $a_{k+1}, \dots, a_n \in N$. By $I(\varphi)$

$$a_{k+1} a_{k+2} \dots a_n - a_{\varphi(k+1)} a_{\varphi(k+2)} \dots a_{\varphi(n)} \in A(b) .$$

It follows that this element lies in $A(N')$.

Though very simple, the above result seems to be useful. E.g., in a near-ring without nilpotent elements, $I(\varphi)$ implies the identity considered there. As a very special case, a regular near-ring satisfying the identity $xyzt = xzyt$ satisfies also the $xyz = yxz$. A much more general result will be shown in Thm. 2.6.

2 - Main results

The following theorem is analogous to $[2]_2$, Thm. 7.

Theorem 2.1. *If the distributive near-ring N satisfies $I(\varphi)$, then the nilpotent elements of N form an ideal J of N, and $J = \mathcal{N}(N)$ (the nil radical of N).*

Proof. Let J be the set of all nilpotent elements of N, let a, $b \in J$. Then there is a positive integer m such that $a^m = b^m = 0$.

We get $(a+b)^{2(m+n)} = \Sigma_i z_i$, where z_i is a product of $2(m+n)$ elements of $\{a, b\}$. For a fixed i, there are at least m factors equal to a (or else m equal to b) in this product, and lying after the first n and before the last n factors. By Lemma 1.1, $z_i = xa^m y$ or $z_i = xb^m y$ thus $z_i = 0$. This proves that $a + b \in J$. The rest is clear, by Lemma 1.1.

Corollary 2.2. *If the distributive near-ring N satisfies $I(\varphi)$, then $N/\mathcal{N}(N)$ is a subdirect sum of integral domains.*

Proof. We may see N as a right near-ring, as well as a left. The assertion follows easily from Lemma 1.2 and $[2]_2$, Thm. 9.

Let us now pass to the W-near-rings. Note that *every regular near-ring is a W-near-ring* (if $x = xx'x$ put $x^0 = xx'$). An example of a non-regular W-near-ring is the following.

Let $(G, +)$ be a cyclic group, whose order is not a prime. Let $D = \{x \in G | G \neq \langle x \rangle\}$. For $x, y \in G$ define $xy = y$ if $x \in D$, $xy = 0$ otherwise. Now $N = (G, +, \cdot)$ is a near-ring.

Let $x \mapsto x^0$ be any map from N to itself, such that $x^0 \in D$ for all x and $x^0 = x$ for $x \in D$. It is now easily checked that N is a W-near-ring.

Lemma 2.3. *If the W-near-ring N satisfies $I(\varphi)$, then N has the* IFP.

Proof. Let $a, b, r \in N$, with $ab = 0$. By Lemma 1.1, we have $arb = a^0 \dots a^0 arb^0 \dots b^0 b = a^0 rab = a^0 r0 = 0$.

Of course, the hypothesis of the preceding lemma can be weakened. The same argument may also be used in order to simplify the proof of $[2]_3$, Thm. 13.

Lemma 2.4. *Let N be a W-near-ring. If $A \neq 0$ then N is neutral and has a left identity.*

Proof. Since A is a N-subgroup, for $w \in A \setminus \{0\}$ we get $w^0 \in A \setminus \{0\}$. If $A(w^0) \neq 0$ then from $w^0 \in A$ it follows $w^0 = (w^0)^2 = 0$, a contradiction. Thus $A(w^0) = 0$.

A known argument (see e.g. $[2]_1$, Lemma 5 or $[2]_3$, Lemma 12, where it is used in the proof) shows that w^0 is a left identity. Now for $x, y \in N$, with $A(x) \neq 0$, we get $xw^0 = 0$ and $xy = xw^0 y = 0y$.

Lemma 2.5. *Let N be a subdirectly irreducible 0-symmetric W-near-ring. If it has the* IFP, *then N is simple.*

Proof. Let H be the intersection of all the non-zero ideals of N, let $w \in H \setminus \{0\}$. We get $w^0 \in H$, because H is an N-subgroup.

If $A(w^0) \neq 0$ then $K \neq 0$ and $A \neq 0$, whence $H \subseteq A$. This implies $(w^0)^2 = 0$, a contradiction. Therefore $A(w^0) = 0$ and, since it is idempotent, w^0 is a left identity. From the 0-symmetry of N it follows $N = w^0 N \subseteq H$. Then N is simple.

In the following statements, we will write x, y, z instead of x_1, x_2, x_3.

Theorem 2.6. *Let N be a W-near-ring. The following are equivalent:*

(1) *$I(\varphi)$ holds in N* (2) *$xyz = yxz$ holds in N*

(3) *N is a subdirect sum of neutral subdirectly irreducible near-rings with $xyz = yxz$.*

Proof. It is clear that $(3) \Rightarrow (2) \Rightarrow (1)$. Only $(1) \Rightarrow (3)$ needs to be proved. Since the image inverse of a $f(N)$-subgroup is a N-subgroup, for each homomorphism f defined on N, it is easy to see that each homomorphic image of N is again a W-near-ring. Then N is a subdirect product of subdirectly irreducible W-near-rings N_i.

Each N_i is neutral (Lemma 2.4). By using $I(\varphi)$ and a left identity of N_i (Lemma 2.4), a straightforward computation yields $xyz = yxz$. It follows (3).

Corollary 2.7. *Let N be a 0-symmetric W-near-ring. The following are equivalent:*

(1) *$I(\varphi)$ holds in N* (2) *$xyz = yxz$ holds in N*

(3) *N is a subdirect sum of integral, N-simple near-rings, each of which satisfies $xyz = yxz$.*

Proof. By Theorem 2.6, we need only to prove that if N is subdirectly irreducible, it is integral. Since N is simple, by Lemma 2.5, and has the IFP, by Lemma 2.3, for $x \in N \setminus \{0\}$ we get $A(x) = 0$ or $A(x) = N$. Clearly $x^0 \notin A(x)$, then $A(x) = 0$.

References

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Riassunto

Proseguiamo qui la nostra indagine sui quasi-anelli soddisfacenti identità semigruppali. Mostriamo, in particolare, che per una classe di quasi-anelli, comprendente tutti quelli regolari, una tale identità implica sempre la commutatività debole (cioè l'identità $xyz = yxz$*).*

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