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Geometric meaning of some equalities concerning the curvature tensor (**)

1. Introduction

The aim of this paper is to show how the notion of bisectional curvature can be used to give geometric meaning to some classical identities and to some known conditions for the Riemann curvature tensor.

More explicitly, we are able to translate the mentioned relations in terms of bisectional curvatures of oriented planes. The new relations do not depend on any choice of bases in the planes.

We begin in 2 with some algebraic remarks, concerning a general quadrilinear mapping on a real vector space (Propositions 1, 2).

In 4 we give a system of equations, involving the bisectional curvatures, that results to be equivalent to a system of classical identities for the Riemann tensor R (Theorem 1). In particular, to give a geometric form to the first Bianchi identity, the triples of mutually orthogonal oriented planes, having a common line, play an essential role.

In Theorem 2 of 7 we give three geometric conditions, which are shown to be equivalent to some almost complex conditions, that often occur in the literature on the almost hermitian manifolds. The second condition involves a special systems of antiholomorphic planes, introduced in my paper $[4]_2$.

Theorem 3 of 8 gives a geometric meaning to another almost complex condition, introduced in my recent paper $[4]_4$. Here again the above mentioned triples of planes are an essential tool.

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Finally, in **9** a well-known curvature identity on the hermitian manifolds, obtained by A. Gray in 1975, is traslated into a relation, involving only geometric invariants of oriented planes (Theorem 4).

The results of the present paper suggest that other relations, concerning the Riemann curvature tensor, could be replaced by geometric relations in terms of bisectional curvature.

2 - Algebraic remarks

Let M be a *riemannian manifold* of dimension $n \geq 4$ and of class C^∞. Let g be the metric of M. All the tensor fields occurring in the paper are assumed to be of class C^∞.

Let x be a point of M and T_x the tangent space to M at x (1).

Two remarks play an essential role in what follows. Consider a real valued *quadrilinear mapping* Q on T_x. Then

Proposition 1. *If equations*

$$(1)\qquad Q(X, Y, Z, W) = -Q(Y, X, Z, W) \qquad Q(X, Y, Z, W) = -Q(X, Y, W, Z)$$

$$(2)\qquad Q(X, Y, Z, W) = Q(Z, W, X, Y)$$

are satisfied by two arbitrary pairs $\tilde{X}$, $\tilde{Y}$ *and* $\tilde{Z}$, $\tilde{W}$ *of vectors of* T_x, *each formed of independent vectors, then equations* (1), (2) *are satisfied by any set* X, Y, Z, W *of vectors of* T_x.

Proposition 2. *If equations* (1), (2) *are satisfied by two arbitrary pairs* $\tilde{X}$, $\tilde{Y}$ *and* $\tilde{Z}$, $\tilde{W}$ *of vectors of* T_x, *each formed of independent vectors, and also equation*

$$(3)\qquad Q(X, Y, Z, W) + Q(X, Z, W, Y) + Q(X, W, Y, Z) = 0$$

is satisfied by any set $\overline{X}$, $\overline{Y}$, $\overline{Z}$, $\overline{W}$ *of orthonormal vectors of* T_x, *then equation* (5) *is satisfied by any set* X, Y, Z, W *of vectors of* T_x

(1) For general references see S. Kobayashi - K. Nomizu [3].

Proposition 1 shows that equations (1), (2) are significant only when any pair X, Y Z, W is formed of *independent* vectors of T_x.

Proposition 2 shows that the system of equations (1), (2), (3) is significant only when any pair X, Y Z, W, occurring in (1), (2), is formed of *independent* vectors of T_x and when the four vectors, occurring in (3), are assumed to be *orthonormal* vectors of T_x.

3 - Proofs

Let B be a real valued *bilinear mapping* on T_x and S be a real valued *quadri linear mapping* on T_x.

Two elementary lemmas are useful

Lemma 1. *If equation*

$$B(X, Y)=0 \tag{4}$$

is satisfied by any pair $\widetilde{X}, \widetilde{Y}$ *of independent vectors of* T_x, *then equation* (4) *is satisfied by any pair* X, Y *of vectors of* T_x.

Lemma 2. *If* S *is skew-symmetric and equation*

$$S(X, Y, Z, W)=0 \tag{5}$$

is satisfied by any set $\overline{X}, \overline{Y}, \overline{Z}, \overline{W}$ *of orthonormal vectors, then equation* (5) *is satisfied by any set* X, Y, Z, W *of vectors of* T_x.

The proof of Lemma 1 is trivial. The proof of Lemma 2 is also easy. If X, Y, Z, W are not independent vectors, then equation (5) is satisfied ([1], Cor. 1, p. 300). If X, Y, Z, W are independent vectors, construct the orthonormal vectors $\overline{X}, \overline{Y}, \overline{Z}, \overline{W}$, choosing $\overline{X}$ in the line spanned by X, $\overline{Y}$ in the plane spanned by X, Y, and so on. Since $S(X, Y, Z, W)$ is equal to $S(\overline{X}, \overline{Y}, \overline{Z}, \overline{W})$ up to a factor, we are able to complete the proof.

To prove Proposition 1, consider first the expression

$$B(X, Y, \widetilde{Z}, \widetilde{W})=Q(X, Y, \widetilde{Z}, \widetilde{W})+Q(Y, X, \widetilde{Z}, \widetilde{W})$$

where $\widetilde{Z}$, $\widetilde{W}$ is an arbitrary pair of independent vectors of T_x. Since B is a bilinear mapping in the first two variables, using Lemma 1, we get $B(X, Y, \widetilde{Z}, \widetilde{W}) = 0$ for any X, Y of T_x. In particular, if X, Y are *dependent* vectors, we derive $Q(X, Y, \widetilde{Z}, \widetilde{W}) = 0$.

Similarly we get $B(\widetilde{X}, \widetilde{Y}, Z, W) = 0$ for any Z, W of T_x and for any pair $\widetilde{X}$, $\widetilde{Y}$ of independent vectors of T_x. In particular, if Z, W are *dependent* vectors, we derive $Q(\widetilde{X}, \widetilde{Y}, Z, W) = 0$.

Now, since for any pair Z, W of dependennt vectors Q is a bilinear form in X, Y, using again Lemma 1, we get $Q(X, Y, Z, W) = 0$ for two arbitrary pairs X, Y Z, W, each formed of *dependent vectors*.

It is now immediate to check that (1) holds true for any vectors X, Y, Z, W of T_x.

As a consequence of (1) the two members of (2) vanish, when X, Y or Z, W are dependent vectors of T_x. This completes the proof of Proposition 1.

To prove Proposition 2, consider the expression

$$S(X, Y, Z, W) = Q(X, Y, Z, W) + Q(X, Z, W, Y) + Q(X, W, Y, Z).$$

By virtue of Proposition 1 equations (1), (2) hold true for any X, Y, Z, W of T_x. We check easily that S is a skew-symmetric quadrilinear form, so Lemma 2 leads to the conclusion.

4 - The classical identities for the Riemann tensor

It is well-known that the *Riemann tensor* R of $T_x^* \otimes T_x^* \otimes T_x^* \otimes T_x^*$ satisfies equations (1), (2), (3) for any X, Y, Z, W of T_x.

On the other hand the *bisectional curvature* of two oriented planes p, q of T_x, defined by the vectors X, Y and Z, W respectively, is given by

$$\chi_{pq} = R(X, Y, Z, W) \begin{vmatrix} X.X & X.Y \\ Y.X & Y.Y \end{vmatrix}^{-\frac{1}{2}} \begin{vmatrix} Z.Z & Z.W \\ W.Z & W.W \end{vmatrix}^{-\frac{1}{2}} \tag{6}$$

where the dot denotes the inner product with respect to the metric g of M and the square roots are assumed to be positive.

We denote by p', q' the same planes as p, q *with opposite orientation*. We call *strictly orthogonal* two planes p, q of T_x, when any line of p is orthogonal to any line of q.

It is useful to introduce *triples* p_1, p_2, p_3 of *mutually orthogonal oriented planes of* T_x, *having a common line*. Any such triple defines a 4-dimensional oriented subspace of T_x. Denote by $\bar{p}_i$ $(i=1, 2, 3)$ the plane, which is strictly orthogonal to p_i in the mentioned 4-space and orient $\bar{p}_i$ in such a way that the oriented bases of p_i, $\bar{p}_i$ give to the 4-space the due orientation.

We are now able to *translate* the system of identities (1), (2), (3) for the Riemann tensor R at point x in terms of bisectional curvature. More explicitly, we will prove

Theorem 1. *The system of equations*

(7) $$\chi_{pq} = -\chi_{p'q} \qquad \chi_{pq} = -\chi_{pq'}$$

(8) $$\chi_{pq} = \chi_{qp} .$$

(9) $$\chi_{p_1\bar{p}_1} + \chi_{p_2\bar{p}_2} + x_{p_3\bar{p}_3} = 0$$

where p, q *are arbitrary oriented planes of* T_x *and* p_1, p_2, p_3 *is any triple of mutually orthogonal oriented planes of* T_x, *having a common line, results to be equivalent to the system of identities* (1), (2), (3) *for the Riemann tensor* R.

Theorem 1 shows that the notion of bisectional curvature permits us to give a *geometric meaning* to the classical identities (1), (2), (3) for the Riemann tensor R.

5 - Proof

Let p, q be arbitrary oriented planes of T_x. Consider two bases $\widetilde{X}\widetilde{Y}$ and $\widetilde{Z}\widetilde{W}$ in p, q respectively; thus $\widetilde{Y}\widetilde{X}$, $\widetilde{W}\widetilde{Z}$ are bases of p', q' respectively.

Choose a unit vector $\overline{X}$ of the line, which is common to the oriented planes p_1, p_2, p_3, occurring in Theorem 1. Let $\overline{X}\,\overline{Y}$, $\overline{X}\,\overline{Z}$, $\overline{X}\,\overline{W}$ be orthonormal bases in these planes. The vectors $\overline{X}$, $\overline{Y}$, $\overline{Z}$, $\overline{W}$ define a 4-dimensional oriented subspace of T_x.

It is easy to check that the couples $\overline{Z}\,\overline{W}$, $\overline{W}\,\overline{Y}$, $\overline{Y}\,\overline{Z}$ are orthonormal bases of the oriented planes $\bar{p}_1, \bar{p}_2, \bar{p}_3$ of 4.

Now, starting from the classical identities (1), (2), (3), concerning the Riemann tensor R, and taking into account the definition (6) of bisectional curvature, we define immediately obtain relations (7), (8), (9).

Conversely, consider first two arbitrary pairs $\tilde{X}\tilde{Y}$, $\tilde{Z}\tilde{W}$ of vectors of T_x, each of them formed by independent vectors. Let $\bar{X}$, $\bar{Y}$, $\bar{Z}$, $\bar{W}$ be any set of four orthonormal vectors of T_x.

Denote by p, q the oriented planes defined by $\tilde{X}\tilde{Y}$ and $\tilde{Z}\tilde{W}$, respectively. Denote then by p_1, p_2, p_3 the oriented planes defined by the couples $\bar{X}\,\bar{Y}$, $\bar{X}\,\bar{Z}$, $\bar{X}\,\bar{W}$, respectively.

We are now able to use equations (7), (8), (9). The definition (6) of bisectional curvature and Proposition 2 of 2 lead us to the classical identities (1), (2), (3) for the Riemann tensor R.

So Theorem 1 is completely proved.

6 - Almost hermitian manifolds

From now on M is assumed to be an *almost hermitian manifold* of dimension $n = 2m \geqslant 4$. Let J be the almost complex structure of M.

For any oriented plane p of the tangent space T_x, the *holomorphic deviation* δ_p $(0 \leqslant \delta_p \leqslant \pi)$ (G. B. Rizza [4]$_1$) is defined by

$$\cos\delta_p = (JX.Y) \begin{vmatrix} X.X & X.Y \\ Y.X & Y.Y \end{vmatrix}^{-\frac{1}{2}} \tag{10}$$

where X, Y is a basis of p, the dot denotes inner product and the square root is assumed to be positive.

Denote by Jp the oriented plane of T_x spanned by JX, JY. Then p is *holomorphic*, if and only if p is orthogonal to Jp. A holomorphic plane p of T_x is said canonically oriented if p is oriented by a couple Z, JZ of vectors of T_x. Finally, we have $\delta_p = 0$, if and only if p is a canonically oriented holomorphic plane. We have $\delta_p = \frac{\pi}{2}$, if and only if p is an antiholomorphic plane.

If p is a non holomorphic plane of T_x, we consider the system $\overset{*}{\sum}_p$ of the ∞^1 antiholomorphic oriented planes of T_x, having a line in common with p and a line in common with Jp (G. B. Rizza [4]$_2$, p. 40).

7 - Almost complex conditions

Many conditions, involving the almost complex structure J and concerning the curvature tensor R, have been considered in the literature.

We begin here to list the *conditions*

$$R(X, Y, Z, W) = R(X, Y, JZ, JW) \tag{11}$$

$$\begin{aligned} &R(X, Y, Z, W) \\ &= R(JX, JY, Z, W) + R(JX, Y, JZ, W) + R(JX, Y, Z, JW) \end{aligned} \tag{12}$$

$$R(X, Y, Z, W) = R(JX, JY, JZ, JW) \tag{13}$$

where X, Y, Z, W are arbitrary vectors of T_x.

For these conditions see for example A. Gray [2], p. 605 and F. Tricerri-L. Vanhecke [6], p. 368.

It is easy to check that (11) implies (12) and that (12) implies (13).

Condition (11) is sometimes called the *Kähler condition* for the curvature tensor. If this condition holds true at any point x of M, then M is called a *parakähler manifold* (G. B. Rizza $[4]_3$) or an *F-space* (S. Sawaki [5]). The Kähler manifolds are a special case.

Condition (13) means that R is J-invariant at the point x of M. An almost hermitian manifold M with J-invariant curvature tensor R at any point is sometimes called an *RK-manifold* (L. Vanhecke [7]).

Condition (12) can be written in the *symmetric form*

$$\begin{aligned} &R(JX, Y, Z, W) + R(X, JY, Z, W) \\ &+ R(X, Y, JZ, W) + R(X, Y, Z, JW) = 0\,. \end{aligned} \tag{14}$$

Now we will prove

Theorem 2. *The conditions*

$$\chi_{pq} = \chi_{pJq} \tag{15}$$

$$(\chi_{pq^*} - \chi_{pJq^*}) \sin \delta_q + (\chi_{p^*q} - \chi_{Jp^*q}) \sin \delta_p = 0 \tag{16}$$

$$\chi_{pq} = \chi_{JpJq} \tag{17}$$

where p, q are arbitrary planes of T_x and p^*, q^* are arbitrary planes of $\overset{*}{\sum}_p$, $\overset{*}{\sum}_q$, result to be equivalent to the conditions (11), (12), (13), *respectively.*

Theorem 2 shows that the notion of bisectional curvature permit us to give a *geometric meaning* to the known almost complex conditions (11), (12), (13) for the curvature tensor R.

We begin to prove that (11) and (15) are equivalent conditions. Given two oriented planes p, q of T_x, let X, Y and Z, W be orthonormal bases for them. Taking account of (6), from (11) we immediately derive (15).

Conversely, since R satisfies (3) of 3, condition (11) is trivially satisfied, when X, Y or Z, W are not independent vectors. Assume then that X, Y and Z, W are two pairs of vectors of T_x, each formed of independent vectors, and denote by p, q, respectively, the oriented planes spanned by X, Y and Z, W. Starting from (15), we imediately prove (11).

Similarly we can prove that condition (13) is equivalent to condition (17).

Since (12), (14) are equivalent conditions, we have only to prove that (14) and (16) are equivalent. Let p, q be two arbitrary oriented planes of T_x and let p^*, q^* be arbitrary oriented planes of $\overset{*}{\sum}_p$, $\overset{*}{\sum}_q$, respectively. We can choose orthonormal bases $\widetilde{X}$, $\widetilde{Y}$ and $\widetilde{Z}$, $\widetilde{W}$ for p, q, such that $\widetilde{X}$, $J\widetilde{Y}$ and $\widetilde{Z}$, $J\widetilde{W}$ span the antiholomorphic oriented planes p^*, q^*. Then, using definitions (6), (10) of 4, 6, from (14) we derive (16).

Conversely, remark first that (14) is trivially satisfied, when X, Y or Z, W are not independent vectors of T_x. Consider then the case, when both pairs X, Y and Z, W are formed of independent vectors of T_x and put

$$X = a\widetilde{X}, \qquad Y = b\widetilde{X} + c\widetilde{Y}, \qquad Z = d\widetilde{Z}, \qquad W = e\widetilde{Z} + f\widetilde{W}$$

where $\widetilde{X}$, $\widetilde{Y}$ and $\widetilde{Z}$, $\widetilde{W}$ are two pairs of orthonormal vectors. Using linearity, we see that condition (14) for X, Y, Z, W is an immediate consequence of condition (14) for $\widetilde{X}$, $\widetilde{Y}$, $\widetilde{Z}$, $\widetilde{W}$.

Finally, let p, q be the oriented planes of T_x spanned by $\widetilde{X}$, $\widetilde{Y}$ and by $\widetilde{Z}$, $\widetilde{W}$. Denote then by p^*, q^* the antiholomorphic oriented planes spanned by $\widetilde{X}$, $J\widetilde{Y}$ and by $\widetilde{Z}$, $J\widetilde{W}$. It is easy now, starting from (16), to derive (14) for $\widetilde{X}$, $\widetilde{Y}$, $\widetilde{Z}$, $\widetilde{W}$.

So condition (14), (16) are equivalent.

This concludes the proof of Theorem 2.

8 - A Bianchi-type condition

In a recent paper of mine $[4]_4$, the *condition*

$$\begin{aligned} &R(X, Y, JZ, JW) + R(X, Z, JW, JY) + R(X, W, JY, JZ) \\ &+ R(JX, JY, Z, W) + R(JX, JZ, W, Y) + R(JX, JW, Y, Z) = 0 \end{aligned} \tag{18}$$

where X, Y, Z, W are assumed to be arbitrary vectors of T_x, is introduced.

Condition (18), also involving the almost complex structure J of the manifold, can be regarded as a *Bianchi-type condition.* Namely, if the curvature tensor R satisfies the Kähler condition (11) of Sec. 7, then condition (18) simply reduces to the classical first Bianchi identity (3).

Using the same notations as in Theorem 1, we will prove

Theorem 3. *The condition*

$$\chi_{p_1 J\bar{p}_1} + \chi_{p_2 J\bar{p}_2} + \chi_{p_3 J\bar{p}_3} + \chi_{Jp_1\bar{p}_1} + \chi_{Jp_2\bar{p}_2} + \chi_{Jp_3\bar{p}_3} = 0 \tag{19}$$

for any triple p_1, p_2, p_3 of mutually orthogonal oriented planes of T_x, having a common line, results to be equivalent to the condition (18).

In other words we can say that the notion of bisectional curvature permit us to give a *geometric meaning* to condition (18), concerning the curvature tensor R.

As we saw in 5, given the oriented planes p_1, p_2, p_3 of T_x, we can choose for them the orthonormal bases $\overline{X}\,\overline{Y}$, $\overline{X}\,\overline{Z}$, $\overline{X}\,\overline{W}$ and check that $\overline{Z}\,\overline{W}$, $\overline{W}\,\overline{Y}$, $\overline{Y}\,\overline{Z}$ are orthonormal bases for $\bar{p}_1$, $\bar{p}_2$, $\bar{p}_3$. Then, using definition (6) of 4, from (18) we immediately derive (19). To prove the converse denote by $S(X, Y, Z, W)$ the first member of (18). It is easy to check that $S(X, Y, Z, W)$ vanishes, if two of the vectors X, Y, Z, W coincide. Therefore S is a skew-symmetric quadrilinear map. Consider now four arbitrary orthonormal vectors $\overline{X}$, $\overline{Y}$, $\overline{Z}$, $\overline{W}$ of T_x and denote by p_1, p_2, p_3 the oriented planes spanned by $\overline{X}\,\overline{Y}$, $\overline{X}\,\overline{Z}$, $\overline{X}\,\overline{W}$. This implies that the oriented planes $\bar{p}_1$, $\bar{p}_2$, $\bar{p}_3$ are spanned by $\overline{Z}\,\overline{W}$, $\overline{W}\,\overline{Y}$, $\overline{Y}\,\overline{Z}$, respectively. Starting from (19), we immediately derive $S(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) = 0$. Using Lemma 2 of 2, we conclude that $S(X, Y, Z, W) = 0$ for any vectors X, Y, Z, W of T_x. So (19) implies (18) and Theorem 3 is proved.

9 - A Gray's identity

In this Section M is assumed to be a *hermitian manifold.*

In 1975 A. Gray proved the following remarkable *identity*

$$\begin{aligned} &R(X, JY, Z, W) + R(X, Y, JZ, W) + R(X, Y, Z, JW) \\ &+ R(JX, Y, Z, W) - R(X, JY, JZ, JW) \\ &- R(JX, Y, JZ, JW) - R(JX, JY, Z, JW) - R(JX, JY, JZ, W) = 0. \end{aligned} \tag{19}$$

(See [2], (3.2), p. 603).

The following theorem gives *geometric meaning* to the Gray's identity.

Theorem 4. *The relation*

$$(20) \qquad (\chi_{pq^*} - \chi_{Jpq^*} - \chi_{pJq^*} + \chi_{JpJq^*}) \sin \delta_q + (\chi_{p^*q} - \chi_{Jp^*q} - \chi_{p^*Jq} + \chi_{Jp^*Jq}) \sin \delta_p = 0$$

where p, q are arbitrary planes of T_x and p^*, q^* are arbitrary planed of $\overset{*}{\Sigma}_p$, $\overset{*}{\Sigma}_q$, results to be equivalent to the identity (19).

The proof of Theorem 4 is very similar to the proof we gave in 7 to show that conditions (14), (16) are equivalent.

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Sommario

Questo lavoro mostra come la nozione di curvatura bisezionale permetta di dare significato geometrico ad alcune classiche identità ed ad alcune note condizioni per il tensore di Riemann.

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