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Almost Finsler structures and almost symplectic structures on tangent bundles

As it is well-known, the theory of G-structures is a theory to understand geometrical structures of manifolds systematically. That is to say, let M be an n-dimensional C^∞-manifold and G be a linear Lie group of order n. We say that M admits a *G-structure* when and only when M is covered by a system of local coordinate neighbourhoods $\{U_\alpha\}$ such that, in each U_α, there exists a local n-frame $\{Z_i{}^{(\alpha)}\}$ satisfying the following: for $\{Z_i{}^{(\alpha)}\}$ in U_α and $\{Z_i{}^{(\beta)}\}$ in U_β, if $U_\alpha \cap U_\beta \neq \phi$ then $Z_i{}^{(\beta)} = P^j_i Z_j{}^{(\alpha)}$ holds in $U_\alpha \cap U_\beta$ where $(P^j_i) \in G$. In this case, the n-frame $\{Z_i{}^{(\alpha)}\}$ in each U_α is said to be *adapted to the G-structure.*

If M admits a G-structure and is covered by a system of local coordinate neighbourhoods $\{(U_\alpha, (x^i{}_{(\alpha)}))\}$ such that, in each U_α, the natural frame $\{\partial/\partial x^i_{(\alpha)}\}$ is adapted to the G-structure, then the G-structure is said to be *integrable.*

On the other hand, on a manifold admitting a G-structure, a linear connection ∇ is called a *G-connection* relative to the G-structure if it satisfies $\nabla_U Z_i = \Gamma^j_{im} U^m Z_j$ where $(\Gamma^j_{im} U^m) \in \mathfrak{g}$ ($\mathfrak{g}$ is the Lie algebra of the Lie group G) for any vector field $U = U^m Z_m$.

It is well-known that, if M admits a G-structure, then there always exists a G-connection relative to the G-structure [3], [11], [20].

As examples of G-structures, the followings are47ell-known:

M is a Riemann manifold (locally Euclidean)	$\rightleftarrows$	M admits an $O(n)$-structure (integrable)
M is an almost complex manifold (complex)	$\rightleftarrows$	M admits a $GL(n, C)$-structure (integrable)

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M is an almost symplectic manifold (symplectic)	$\rightleftarrows$	M admits an $Sp(n)$-structure (integrable)
M is an almost contact metric manifold (contact metric)	$\rightleftarrows$	M admits a $U(n)\times 1$-structure (with some condition)

Now, from this point of view, we would like to reconsider the theory of Finsler manifolds and to define a notion of an *almost Finsler structure.* Concretely speaking, corresponding to some linear Lie group G of order $2n$, we consider the following problems:

M is an almost Finsler manifold (Finsler)	$\rightleftarrows$	$T(M)$ admits the G-structure (satisfying some condition)

In other words, we would like to survey the

Conjecture. *A manifold M is a Finsler manifold if and only if the tangent bundle $T(M)$ admits a G-structure satisfying some condition.*

To solve this conjecture, we must find, in the first place, the linear Lie group G, and next, we must define a notion of an almost Finsler structure, then we will prove the conjecture.

The main purpose of the present paper is, first of all, to solve this conjecture. In this case, an almost symplectic structure defined on the tangent bundle actually plays an important role. So, the almost Hamilton vector fields associated with the almost symplectic structure is investigated. The G-connection relative to the almost Finsler structure is also dealt with. Moreover we discuss the case when a non-linear connection is assigned. The integrability conditions of these structures are also investigated, from which the situation of the locally Minkowski manifolds becomes clear.

Throughout the paper, we use the following indices and notations:

$A, B, C, \dots, P, Q, R, \dots$ run over the range $\{1, 2, 3, \dots, 2n\}$

$a, b, c, \dots, i, j, k, \dots$ run over the range $\{1, 2, 3, \dots, n\}$

$\bar{a}, \bar{b}, \dots, \bar{i}, \bar{j}, \dots$ stand for $a+n, b+n, \dots, i+n, j+n, \dots$ respectively.

With respect to any canonical coordinate system in a tangent bundle, we write $(x^A)=(x^a, x^{\bar{a}})=(x^a, y^a)$, ∂_i and $\dot{\partial}_i$ stand for $\partial/\partial x^i$ and $\partial/\partial y^i$ respectively.

1 - The Finsler group

Let M be an n-dimensional C^∞-manifold and $T(M)$ be its tangent bundle. As is well-known, $T(M)$ admits a standard integrable almost tangent structure, that is, standard tangent structure $\mathscr{F}_0$ ([1], [2], [3], [5], [19]). Let Q be its structure tensor.

Now $T(n)=\{\begin{pmatrix} A & 0 \\ B & A \end{pmatrix} \mid A \in GL(n, R),\ B \in gl(n, R)\}$ is a well-known Lie group, which we call an *n-dimensional tangent group*. The structure $\mathscr{F}_0$ is a $T(n)$-structure on $T(M)$ and is integrable. Let π be the natural projection $T(M) \to M$. Any coordinate neighbourhood (U, x^i) in M induces a coordinate neighbourhood $(\pi^{-1}(U), (x^i, y^i))$ in $T(M)$, which we call a *canonical coordinate neighbourhood.* For two arbitrary canonical coordinate neighbourhoods $(\pi^{-1}(U), (x^i, y^i))$, $(\pi^{-1}(\bar U), (\bar x^i, \bar y^i))$ such that $\pi^{-1}(U) \cap \pi^{-1}(\bar U) \neq \phi$, $2n$-frames $\{Z_A\}$ in $\pi^{-1}(U)$ and $\{\bar Z_A\}$ in $\pi^{-1}(\bar U)$, both adapted to the $T(n)$-structure, satisfy $\bar Z_A = P^B{}_A Z_B$ $((P^B{}_A) \in T(n))$ in $\pi^{-1}(U) \cap \pi^{-1}(\bar U)$. Of course, the components $a^i{}_j$ and
$b^i{}_j$ of $(P^B{}_A) = \begin{pmatrix} a^i{}_j & 0 \\ b^i{}_j & a^i{}_j \end{pmatrix}$ are functions of x^i and y^i.

Now, we may consider the case where $a^i{}_j$ are *positively homogeneous of degree* 0 with respect to y^i and $b^i{}_j$ are *positively homogeneous of degree* 1 with respect to y^i. This is well-defined by virtue of the transformation law of (x^i, y^i) and the form of $\begin{pmatrix} A_1 & 0 \\ B_1 & A_1 \end{pmatrix}\begin{pmatrix} A_2 & 0 \\ B_2 & A_2 \end{pmatrix} = \begin{pmatrix} A_1A_2 & 0 \\ B_1A_2 + A_1B_2 & A_1A_2 \end{pmatrix}$. If we treat only the structure $\mathscr{F}_0$ under the restriction of the above homogeneity condition, we call it a *homogeneous standard tangent structure* and denote it by $\mathscr{F}_0^*$. The condition for $2n$-frames $\{Z_A\}$ to be adapted to the structure $\mathscr{F}_0^*$ is that Z_a and $Z_{\bar a}$ can be written as

$$Z_a = \gamma^i{}_a \frac{\partial}{\partial x^i} + \gamma^{\bar i}{}_a \frac{\partial}{\partial y^i} \qquad Z_{\bar a} = \gamma^i{}_a \frac{\partial}{\partial y^i} \qquad (\det|\gamma^i{}_a| \neq 0)$$

in any canonical coordinate neighbourhood $(\pi^{-1}(U), (x^i, y^i))$ and $\gamma^{\bar i}{}_a$ are *positively homogeneous of degree* 1 with respect to y^i and $\gamma^i{}_a$ are *positively homogeneous of degree* 0 with respect to y^i.

Now, let G be a Lie subgroup of $T(n)$. If $T(M)$ admits the G-structure $\mathscr{B}$ and if any frame adapted to the structure $\mathscr{B}$ is always adapted to the standard tangent structure $\mathscr{F}_0$, ($\mathscr{B}$ is the G-structure as a reduction of $\mathscr{F}_0$), then the structure $\mathscr{B}$ is called a *G-structure depending on* $\mathscr{F}_0$. The condition for $\mathscr{B}$ to be a G-structure depending on $\mathscr{F}_0$ is given as follows:

(1) G is a Lie subgroup of $T(n)$.

(2) For any canonical coordinate neighbourhoods $\pi^{-1}(U)$ and $\pi^{-1}(\bar{U})$ such that $\pi^{-1}(U)\cap\pi^{-1}(\bar{U})\neq\phi$, there exists $2n$-frames $\{Z_A\}$ in $\pi^{-1}(U)$ and $\{\bar{Z}_A\}$ in $\pi^{-1}(\bar{U})$ satisfying $\bar{Z}_A=P^B{}_A Z_B$ $((P^B{}_A)\in G)$ in $\pi^{-1}(U)\cap\pi^{-1}(\bar{U})$.

(3) In each $\pi^{-1}(U)$, the above $2n$-frame $\{Z_A\}$ adapted to $\mathscr{B}$ is written as

$$Z_a=\gamma^i{}_a\frac{\partial}{\partial x^i}+\gamma^{\bar{i}}{}_a\frac{\partial}{\partial y^i}\qquad Z_{\bar{a}}=\gamma^i{}_a\frac{\partial}{\partial y^i}\qquad \text{where } \det|\gamma^i{}_a|\neq 0.$$

In the case when we treat the homogeneous standard tangent structure $\mathscr{F}_0^*$, we must add the condition that $P^i{}_j$ and $\gamma^i{}_j$ are positively homogeneous of degree 0 with respect to y^i and $P^{\bar{i}}{}_j$ and $\gamma^{\bar{i}}{}_j$ are positively homogeneous of degree 1 with respect to y^i where we put $(P^B{}_A)=\begin{pmatrix}P^i{}_j & 0\\ P^{\bar{i}}{}_j & P^i{}_j\end{pmatrix}$. The G-structure $\mathscr{B}$ under the restriction of the above homogeneity condition is called a *homogeneous G-structure depending on* $\mathscr{F}_0^*$, and is denoted by $\mathscr{B}^*$ (1).

Now let us consider the following set

$$F(n)=\{\begin{pmatrix}A & 0\\ SA & A\end{pmatrix}\ |A\in O(n),\ S\in \mathrm{Sym}(n)\}\ (2)$$

For any $\begin{pmatrix}A_1 & 0\\ S_1A_1 & A_1\end{pmatrix}$, $\begin{pmatrix}A_2 & 0\\ S_2A_2 & A_2\end{pmatrix}\in F(n)$, we see that

$$\begin{pmatrix}A_1 & 0\\ S_1A_1 & A_1\end{pmatrix}\begin{pmatrix}A_2 & 0\\ S_2A_2 & A_2\end{pmatrix}=\begin{pmatrix}A_1A_2 & 0\\ (S_1+A_1S_2{}^tA_1)A_1A_2 & A_1A_2\end{pmatrix}$$

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(1) This homogeneity condition can be written as $d\tilde{\lambda}(Z_a)_{(x,y)}=(Z_a)_{(x,\lambda y)}$ where $d\tilde{\lambda}$ is the differential of the mapping $\tilde{\lambda}$: $T(M)\to T(M)\,((x,\ y)\to(x,\ \lambda y))$, λ being any positive number.

(2) $F(n)$ is similar to the tangent orthogonal group $T(O(n))$ defined by Morimoto [15]. But $F(n)$ does not coincide with $T(O(n))$. In $T(O(n))$, S is an element of the Lie algebra $\mathfrak{o}(n)$ of $O(n)$.

and $S_1 + A_1 S_2 {}^tA_1 \in \mathrm{Sym}(n)$. Moreover we see

$$\begin{pmatrix} A_1 & 0 \\ S_1 A_1 & A_1 \end{pmatrix}^{-1} = \begin{pmatrix} {}^tA_1 & 0 \\ -{}^tA_1 S_1 A_1 {}^tA_1 & {}^tA_1 \end{pmatrix}$$

and $-{}^tA_1 S_1 A_1 \in \mathrm{Sym}(n)$. Hence, it is apparent that $F(n)$ is a Lie subgroup of $T(n)$. Hereafter the Lie group $F(n)$ is called a *Finsler group*, an $F(n)$-structure depending on $\mathscr{F}_0$ is called an *almost Finsler structure* and is denoted by $\mathscr{F}_1$. Similarly, a homogeneous $F(n)$-structure depending on $\mathscr{F}_0^*$ is called a *homogeneous almost Finsler structure* and is denoted by $\mathscr{F}_1^*$ [6], [8], [9].

If we put $J = (J_{PQ}) = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}$, we easily see that any $P \in F(n)$ satisfies ${}^tPJP = J$, that is, $F(n) \subset Sp(n)$ ($Sp(n)$ is the n-dimensional symplectic group). Also we can easily see that $T(n) \cap Sp(n) = F(n)$. From the theory of G-structures [3], we obtain

Theorem 1.1. *If a tangent bundle $T(M)$ admits an almost Finsler structure, $T(M)$ also admits an almost symplectic structure. And $T(n) \cap Sp(n) = F(n)$ holds good.*

On a tangent bundle, the canonical frame $\{\frac{\partial}{\partial x^A}\}$ is adapted to $\mathscr{F}_0$. On the other hand, if $\{Z_A\}$ is adapted to $\mathscr{F}_1$, then is adapted to $\mathscr{F}_0$. Hence in each $\pi^{-1}(U)$, we have $Z_A = \gamma^B{}_A \partial_B$, $((\gamma^B{}_A) = \Gamma \in T(n))$ for $2n$-frame $\{Z_A\}$ adapted to $\mathscr{F}_1$, also we have $\partial_A = \beta^B{}_A Z_B$ where $B = (\beta^B{}_A) = \Gamma^{-1} \in T(n)$. Now, as is well-known [3], $\omega_{AB} = J_{PQ}\beta^P{}_A \beta^Q{}_B$ becomes a global tensor field on $T(M)$ and satisfies $\det|\omega_{AB}| \neq 0$, $\omega_{AB} = -\omega_{BA}$. That is, $\Omega = \omega_{AB}\, dx^A \wedge dx^B$ *is just the 2-form associated with the almost symplectic structure under consideration.*

Next, we can define a *pseudo inner product* $\langle , \rangle$ of rank n by $\langle Z_i, Z_j \rangle = \delta_{ij}$, $\langle Z_i, Z_{\bar{j}} \rangle = \langle Z_{\bar{i}}, Z_j \rangle = \langle Z_{\bar{i}}, Z_{\bar{j}} \rangle = 0$ in each $\pi^{-(U)}$. Because of the properties of the $F(n)$-structure (i.e., $A \in O(n)$), we can easily see that the above pseudo inner product can be extended to the whole of $T(M)$ globally. Thus *we obtain a singular Riemann metric G' of rank n on $T(M)$.* In each $\pi^{-1}(U)$, we have $G'_{AB} = \langle \frac{\partial}{\partial x^A}, \frac{\partial}{\partial x^B} \rangle = \sum_{a=1}^{n} \beta^a{}_A \beta^a{}_B$. Putting $\sum_{a=1}^{n} \beta^a{}_i \beta^a{}_j = g_{ij}$ and $\sum_{a=1}^{n} (\beta^{\bar{a}}{}_i \beta^a{}_j - \beta^a{}_i \beta^{\bar{a}}{}_j) = \alpha_{ij}$, we obtain

$$(G'_{AB}) = \begin{pmatrix} g_{ij} & 0 \\ 0 & 0 \end{pmatrix} \qquad (\omega_{AB}) = \begin{pmatrix} \alpha_{ij} & -g_{ij} \\ g_{ij} & 0 \end{pmatrix} \qquad \Omega = \alpha_{ij}\, dx^i \wedge dx^j - 2g_{ij}\, dx^i \wedge dy^j .$$

The Lie algebra $\mathfrak{f}(n)$ of the Lie group $F(n)$ is given by

$$\mathfrak{f}(n)=\{\begin{pmatrix} a & 0 \\ b & a\end{pmatrix} \mid a\in O(n),\ b\in \mathrm{Sym}(n)\}.$$

Now, let ∇ be a G-connection relative to the structure $\mathscr{F}_1$ (i.e., ∇ is an $F(n)$-connection), then $\nabla_U Z_A=\tilde{\tilde{\Gamma}}{}^B_{AC}U^C Z_B$ where $\tilde{\tilde{\Gamma}}{}^B_{AC}U^C\in\mathfrak{f}(n)$ in each $\pi^{-1}(U)$. With respect to any $2n$-frame adapted to $\mathscr{F}_1$, we see that

$$G'=(\tilde{\tilde{G}}{}'_{AB})=\begin{pmatrix} E & 0 \\ 0 & 0\end{pmatrix}\qquad \omega=(\tilde{\tilde{\omega}}_{AB})=\begin{pmatrix} 0 & -E \\ E & 0\end{pmatrix}\qquad Q=(\tilde{\tilde{Q}}{}^A{}_B)=\begin{pmatrix} 0 & 0 \\ E & 0\end{pmatrix}.$$

Hence by calculating $\nabla G'$, $\nabla\omega$ and ∇Q, we easily obtain

Theorem 1.2. *Let $T(M)$ be a tangent bundle admitting an almost Finsler structure $\mathscr{F}_1$. In order that a linear connection ∇ on $T(M)$ be a G-connection relative to the structure $\mathscr{F}_1$, it is necessary and sufficient that $\nabla\omega=0$ and $\nabla Q=0$ hold good. With respect to the connection ∇, $\nabla G'=0$ is also true.*

2 - Homogeneous almost Finsler structures

In the following we treat the *homogeneous* almost Finsler structure $\mathscr{F}_1^*$. Consider again the 2-form

$$\Omega=\omega_{AB}\,\mathrm{d}x^A\wedge\mathrm{d}x^B=\alpha_{ij}\,\mathrm{d}x^i\wedge\mathrm{d}x^j-2g_{ij}\,\mathrm{d}x^i\wedge\mathrm{d}y^j. \tag{2.1}$$

In the present case the g_{ij}'s are positively homogeneous of degree 0 with respect to y^i (shortly (0)p-homogeneous for y), and the α_{ij}'s are positively homogeneous of degree 1 with respect to y^i (shortly (1)p-homogeneous for y). So, the g_{ij}'s *give M a generalized metric in the sense of A. Moór* [14].

Next, ω_{AB} is a skew-symmetric tensor field on $T(M)$. So, by direct calculation, in each $\pi^{-1}(U)\cap\pi^{-1}(\bar{U})$ where $U\cap\bar{U}\neq\phi$, we find the following transformation rules for g_{ij} and α_{ij}

$$\begin{aligned} g_{ij}&=\bar{g}_{pq}\frac{\partial\bar{x}^p}{\partial x^i}\frac{\partial\bar{x}^q}{\partial x^j}\\ \alpha_{ij}&=\bar{\alpha}_{pq}\frac{\partial\bar{x}^p}{\partial x^i}\frac{\partial\bar{x}^q}{\partial x^j}-\bar{g}_{pq}\frac{\partial\bar{x}^p}{\partial x^i}\frac{\partial^2\bar{x}^q}{\partial x^j\partial x^m}y^m+\bar{g}_{pq}\frac{\partial^2\bar{x}^p}{\partial x^i\partial x^m}y^m\frac{\partial\bar{x}^q}{\partial x^j}.\end{aligned}\tag{2.2}$$

Thus we obtain

Theorem 2.1. *If a tangent bundle $T(M)$ admits a homogeneous almost Finsler structure, then $T(M)$ admits an almost symplectic structure whose associated 2-form is given by $\Omega = \alpha_{ij}\,dx^i \wedge dx^j - 2g_{ij}\,dx^i \wedge dy^j$. Here, α_{ij} is a quantity such that $\alpha_{ji} = -\alpha_{ij}$ and is positively homogeneous of degree 1 for y^i, g_{ij} is a generalized metric of M, and the transformation rules of α_{ij} are given by* (2.2).

Now, let $\dot{N}$ be a non-linear connection defined on $T(M)$ ([7], [12]) and $\dot{N}^i_j$ be the components of $\dot{N}$ with respect to the canonical local coordinates (x^i, y^i). Then $\dot{N}^i_j$ satisfies the transformation rule

$$\frac{\partial \bar{x}^p}{\partial x^m}\dot{N}^m_j - \bar{\dot{N}}^p_q \frac{\partial \bar{x}^q}{\partial x^j} = \frac{\partial^2 \bar{x}^p}{\partial x^j \partial x^m} y^m .$$

By using this equation, we can show easily that $\beta_{ij} = \alpha_{ij} + g_{im}\dot{N}^m_j - g_{jm}\dot{N}^m_i$ is a skew-symmetric quasi tensor(3) on M [7] and is positively homogeneous of degree 1 for y^i. Hence, if we put $N^i_j = \dot{N}^i_j - \frac{1}{2}g^{im}\beta_{mj}$, then we can show directly that N^i_j gives $T(M)$ a non-linear connection and satisfies $\alpha_{ij} = -g_{im}N^m_j + g_{jm}N^m_i$. Thus we obtain

Theorem 2.2. *Let $g = (g_{ij})$ and $\alpha = (\alpha_{ij})$ be the quantities defined in Theorem 2.1. On $T(M)$, there always exists a non-linear connection N satisfying the condition*

$$\alpha = -gN + {}^tNg . \tag{2.3}$$

Next, let N and $\tilde{N}$ be any two non-linear connections satisfying the condition (2.3). Then $\tilde{N} - N$ is a (1, 1) quasi tensor field on M and is positively homogeneous of degree 1 for y^i. Now, $k = (k_{ij}) = g(\tilde{N} - N)$ is a (0, 2) quasi tensor

(3) A quasi tensor is a so-called tensor of Finsler type. However to avoid the confusion with a Finsler metric tensor, we shall adopt the terminology *quasi tensor*.

field on M, that is positively homogeneous of degree 1 for y^i and satisfies

$$ {}^t k = ({}^t\tilde{N} - {}^t N) g = (\alpha + g\tilde{N}) - (\alpha + gN) = k . $$

So, k is a symmetric quasi tensor field.

Conversely, let N be the non-linear connection shown in Theorem 2.2 and k be any symmetric (0, 2) quasi tensor field and be positively homogeneous of degree 1 for y^i. Then $\tilde{N} = N + g^{-1}k$ satisfies

$$ -g\tilde{N} + {}^t\tilde{N} g = -gN + {}^t N g = \alpha . $$

Thus we obtain

Theorem 2.3. *In a tangent bundle admitting a homogeneous almost Finsler structure, let N be a non-linear connection satisfying the condition* (2.3). *If $\tilde{N}$ is another non-linear connection satisfying the condition* (2.3), *then $\tilde{N}$ is written as $\tilde{N} = N + g^{-1}k$ where k is a* (0, 2) *symmetric quasi tensor field on M and is positively homogeneous of degree* 1 *for y^i. And the converse is also true.*

Now let us consider the converse of Theorem 2.1. That is to say, we assume that a manifold M admits a generalized metric g and a skew-symmetric quantity $\alpha = (\alpha_{ij})$ which positively homogeneous of degree 1 for y^i and satisfies the transformation rule (2.2). In this case, $\Omega = \alpha_{ij}\,dx^i \wedge dx^j - 2g_{ij}\,dx^i \wedge dy^j$ is a globally defined non-degenerate 2-form on $T(M)$. First, we consider a local coordinate neighbourhood (U, x^i). With respect to the generalized metric g, it is easy to find in U n linearly independent local covariant quasi vectors σ_i^a such that $g_{ij} = \sum_{a=1}^{n} \sigma_i^a \sigma_j^a$. That is, $g = {}^t\sigma\sigma$ where $\sigma = (\sigma_i^a)$. Now, we put $\tau = (\tau_a^i) = \sigma^{-1}$. Of course, σ_i^a and τ_a^i are positively homogeneous of degree 0 for y^i. Let N be the non-linear connection shown in Theorem 2.2, i.e., N satisfies $\alpha = -gN + {}^t N g$. Then we can define, on $\pi^{-1}(U)$, a local $2n$-frame $\{Z_A\}$ by

$$ Z_a = \tau_a^i(\partial/\partial x^i - N_i^m \partial/\partial y^m) \qquad Z_{\bar{a}} = \tau_a^i \partial/\partial y^i . $$

The quantities, σ, τ, N and $\{Z_A\}$ always exist on $\pi^{-1}(U)$. However, they can not be determined uniquely. Next, let $(\bar{U}, \bar{x}^i)$ be another local coordinate neighbourhood such that $U \cap \bar{U} \neq \phi$. Then, on $(\pi^{-1}(\bar{U}), (\bar{x}^i, \bar{y}^i))$, we can define similarly σ, τ, N and $\{Z_A\}$, which we denote by $\bar{\lambda}$, $\bar{\mu}$, $\bar{\bar{N}}$ and $\{\tilde{Z}_A\}$ respectively. Now, on

$\pi^{-1}(U)\cap\pi^{-1}(\bar{U})$, we can consider these quantities in terms of the local canonical coordinate system $(x^i,\ y^i)$, which we denote by λ, μ, $\tilde{N}$ and $\{\tilde{Z}_A\}$ respectively. Then, we see

$$\tilde{Z}_a=\mu_a^i(\partial/\partial x^i-\tilde{N}_i^m\,\partial/\partial y^m)\qquad \tilde{Z}_{\bar{a}}=\mu_a^i\,\partial/\partial y^i.$$

Now, in $\pi^{-1}(U)\cap\pi^{-1}(\bar{U})$, $\{Z_A\}$ and $\{\tilde{Z}_A\}$ have, of course, the relation

$$\tilde{Z}_A=P_A^B Z_B\qquad (P_A^B)=\begin{pmatrix}P_a^b & P_{\bar{a}}^b\\ P_a^{\bar{b}} & P_{\bar{a}}^{\bar{b}}\end{pmatrix}\in GL(2n,\ R).$$

First, $Z_{\bar{a}}=P_{\bar{a}}^m Z_m+P_{\bar{a}}^{\bar{m}} Z_{\bar{m}}$ can be rewritten as

$$\mu_a^i\,\partial/\partial y^i=P_{\bar{a}}^m\,\tau_m^i(\partial/\partial x^i-N_i^r\,\partial/\partial y^r)+P_{\bar{a}}^{\bar{m}}\,\tau_m^i\,\partial/\partial y^i.$$

Hence we have $P_{\bar{a}}^m=0$ and $P_{\bar{a}}^{\bar{m}}=\sigma_r^m\mu_a^r$. Secondly, $\tilde{Z}_a=P_a^m Z_m+P_a^{\bar{m}} Z_{\bar{m}}$ can be rewritten as

$$\mu_a^i(\partial/\partial x^i-\tilde{N}_i^m\,\partial/\partial y^m)=P_a^m\,\tau_m^i(\partial/\partial x^i-N_i^r\,\partial/\partial y^r)+P_a^{\bar{m}}\,\tau_m^i\,\partial/\partial y^i.$$

Hence we have $P_a^m=\sigma_r^m\mu_a^r$ and $P_a^{\bar{m}}=\sigma_r^m N_t^r\tau_s^t P_a^s-\sigma_r^m\tilde{N}_t^r\mu_a^t$. Putting $A=(P_a^m)$, we see

$$^tAA={}^t(\sigma\mu)(\sigma\mu)={}^t\mu g\mu={}^t\mu{}^t\lambda\lambda\mu={}^t(\lambda\mu)(\lambda\mu)=E_n$$

i.e., $A\in O(n)$. Next, putting $B=(P_a^{\bar{m}})$, we see, by virtue of Theorem 2.3

$$B=\sigma N\tau A-\sigma\tilde{N}\mu=\sigma N\tau A-\sigma N\tau A-\sigma g^{-1}k\tau A=-\sigma\tau^t\tau k\tau A=-{}^t\tau k\tau A$$

where k is a symmetric matrix. So, putting $S=-{}^t\tau k\tau$, we have ${}^tS=S$, i.e., $S\in\mathrm{Sym}(n)$. Thus we get $(P_B^A)=\begin{pmatrix}A & 0\\ SA & A\end{pmatrix}$ where $A\in O(n)$ and $S\in\mathrm{Sym}(n)$. That is to say, $(P_B^A)\in F(n)$. And, for the relations

$$Z_a=\tau_a^i\,\partial/\partial x^i-\tau_a^i N_i^m\,\partial/\partial y^m\qquad Z_{\bar{a}}=\tau_a^i\,\partial/\partial y^i$$

we have seen already that $\det|\tau_a^i|\neq 0$ and that τ_a^i is positively homogeneous of degree 0 for y^i, and $\tau_a^i N_i^m$ is positively homogeneous of degree 1 for y^i.

Moreover, ${}^t(\tau^{-1})\,\tau^{-1} = {}^t\sigma\sigma = g$ and $\begin{pmatrix} \tau & 0 \\ -N\tau & \tau \end{pmatrix}^{-1} = \begin{pmatrix} \sigma & 0 \\ \sigma N & \sigma \end{pmatrix} = \begin{pmatrix} \sigma^a_i & \sigma^a_{\bar{i}} \\ \sigma^{\bar{a}}_i & \sigma^{\bar{a}}_{\bar{i}} \end{pmatrix}$. So, we have

$$\sum_{a=1}^{n} (\sigma^{\bar{a}}_i \sigma^a_j - \sigma^a_i \sigma^{\bar{a}}_j) = \sum_{a=1}^{n} (\sigma^a_j \sigma^a_m N^m_i - \sigma^a_i \sigma^a_m N^m_j) = g_{mj} N^m_i - g_{im} N^m_j = \alpha_{ij}\,.$$

Thus, as the converse of Theorem 2.1 we obtain

Theorem 2.4. *Assume that a manifold M admits a generalized metric g_{ij} and a skew-symmetric quantity α_{ij} which is positively homogeneous of degree* 1 *for y^i and satisfies the transformation rule* (2.2). *Then $T(M)$ admits a homogeneous almost Finsler structure whose associated almost Finsler 2-form is given by*

$$\alpha_{ij}\, \mathrm{d}x^i \wedge \mathrm{d}x^j - 2 g_{ij}\, \mathrm{d}x^i \wedge \mathrm{d}y^j\,.$$

3 - Finsler structures

Let $T(M)$ be a tangent bundle admitting a homogeneous almost Finsler structure. Applying the exterior differentiation d to the almost Finsler 2-form $\Omega = \alpha_{ij}\, \mathrm{d}x^i \wedge \mathrm{d}x^j - 2g_{ij}\, \mathrm{d}x^i \wedge \mathrm{d}y^j$, we get

$$\mathrm{d}\Omega = \partial_k \alpha_{ij}\, \mathrm{d}x^k \wedge \mathrm{d}x^i \wedge \mathrm{d}x^j + (\dot{\partial}_k \alpha_{ij} + 2\partial_j g_{ik})\, \mathrm{d}y^k \wedge \mathrm{d}x^i \wedge \mathrm{d}x^j - 2\dot{\partial}_k g_{ij}\, \mathrm{d}y^k \wedge \mathrm{d}x^i \wedge \mathrm{d}y^j\,.$$

So, the condition for Ω to be closed can be written as

$$\dot{\partial}_k g_{ij} - \dot{\partial}_j g_{ik} = 0 \qquad \dot{\partial}_k \alpha_{ij} + 2\partial_j g_{ik} - \dot{\partial}_k \alpha_{ji} - 2\partial_i g_{jk} = 0 \qquad \partial_k \alpha_{ij} + \partial_i \alpha_{jk} + \partial_j \alpha_{ki} = 0\,.$$

The first condition means that g_{ij} is a Finsler metric [14]. The second condition leads us to $\dot{\partial}_k \alpha_{ij} = \partial_i g_{jk} - \partial_j g_{ik}$. Since α_{ij} is positively homogeneous of degree 1 for y^i, we obtain

$$\alpha_{ij} = y^m(\partial_i g_{jm} - \partial_j g_{im})\,. \tag{3.1}$$

Conversely, let g_{ij} be a Finsler metric and α_{ij} be the quantity given by (3.1). From the well-known equation $y^m \dot{\partial}_k g_{im} = 0$, we get $\dot{\partial}_k \alpha_{ij} = \partial_i g_{jk} - \partial_j g_{ik}$. Hence, the second condition is clearly satisfied. In this case, moreover, we see

$$\partial_k \alpha_{ij} + \partial_i \alpha_{jk} + \partial_j \alpha_{ki}$$

$$= y^m(\partial_k \partial_i g_{jm} - \partial_k \partial_j g_{im} + \partial_i \partial_j g_{km} - \partial_i \partial_k g_{jm} + \partial_j \partial_k g_{im} - \partial_j \partial_i g_{km}) = 0\,.$$

That is, the third condition is also satisfied. Thus we obtain

Theorem 3.1. *Let $T(M)$ be a tangent bundle admitting a homogeneous almost Finsler structure. The almost Finsler 2-form* $\Omega = \alpha_{ij}\,\mathrm{d}x^i \wedge \mathrm{d}x^j - 2g_{ij}\,\mathrm{d}x^i \wedge \mathrm{d}y^j$ *is closed if and only if g_{ij} is a Finsler metric and α_{ij} is given by* $\alpha_{ij} = y^m(\partial_i g_{jm} - \partial_j g_{im})$.

In the case of Theorem 3.1 we have

$$\Omega = y^m(\partial_i g_{jm} - \partial_j g_{im})\,\mathrm{d}x^i \wedge \mathrm{d}x^j - 2g_{ij}\,\mathrm{d}x^i \wedge \mathrm{d}y^j = \mathrm{d}(2y^m g_{mj}\,\mathrm{d}x^j)\,.$$

That is, Ω is the well-known exact form [**20**]. In the paper [**6**], we have called this Ω *the Finsler form associated with a Finsler metric* and denoted it by Ω^*. Since Ω^* is determined by a Finsler metric only, it seems to us that Theorem 3.1 tells us a new definition and a new treatment of a Finsler manifold.

Next, let there be given a scalar field $\sigma(x, y)$ on $T(M)$, which is positively homogeneous of degree 0 with respect to y. If $T(M)$ admits a homogeneous almost Finsler structure whose associated 2-form is given by $\Omega = \alpha_{ij}\,\mathrm{d}x^i \wedge \mathrm{d}x^j - 2g_{ij}\,\mathrm{d}x^i \wedge \mathrm{d}y^j$, then $T(M)$ also admits another 2-form $\tilde{\Omega} = e^{\sigma(x,y)}\Omega$. Putting $\tilde{g}_{ij} = e^{\sigma(x,y)} g_{ij}$ and $\tilde{\alpha}_{ij} = e^{\sigma(x,y)} \alpha_{ij}$ we have $\tilde{\Omega} = \tilde{\alpha}_{ij}\,\mathrm{d}x^i \wedge \mathrm{d}x^j - 2\tilde{g}_{ij}\,\mathrm{d}x^i \wedge \mathrm{d}y^j$. Of course, $\tilde{g}_{ij}$ is a generalized metric. With respect to $\tilde{\alpha}_{ij}$, it is easy to verify

$$\tilde{\alpha}_{ij} = \bar{\tilde{\alpha}}_{pq} \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial \bar{x}^q}{\partial x^j} - \bar{\tilde{g}}_{pq} \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial^2 \bar{x}^q}{\partial x^j \partial x^m} y^m + \bar{\tilde{g}}_{pq} \frac{\partial \bar{x}^q}{\partial x^j} \frac{\partial^2 \bar{x}^p}{\partial x^j \partial x^i \partial x^m} y^m\,.$$

Thus $T(M)$ admits another homogeneous almost Finsler structure whose associated 2-form is $\tilde{\Omega}$ itself. The condition for $\tilde{\Omega}$ to be closed is given by

(1) $\tilde{g}_{ij}$ is a Finsler metric (2) $\tilde{\alpha}_{ij} = y^m(\partial_i \tilde{g}_{jm} - \partial_j \tilde{g}_{im})$.

The condition (1) implies that the generalized metric g_{ij} is conformal to a Finsler metric. From the condition (2), we have

$$\alpha_{ij} = y^m(\partial_i g_{jm} - \partial_j g_{im}) + \partial_i \sigma g_{jm} y^m - \partial_j \sigma g_{im} y^m . \tag{3.2}$$

Conversely, let $\Omega = \alpha_{ij} \mathrm{d}x^i \wedge \mathrm{d}x^j - 2g_{ij} \mathrm{d}x^i \wedge \mathrm{d}y^j$ be a 2-form on $T(M)$. If there exists such a scalar field $\sigma = \sigma(x, y)$ that $\sigma(x, y)$ is positively homogeneous of degree 0 with respect to y, $e^\sigma g_{ij}$ is a Finsler metric and the relation (3.2) holds, then $e^\sigma \alpha_{ij} = y^m\{\partial_i(e^\sigma g_{jm}) - \partial_j(e^\sigma g_{im})\}$ holds good and $e^\sigma \Omega$ becomes closed. Thus we obtain

Theorem 3.2. *Let $\Omega = \alpha_{ij} \mathrm{d}x^i \wedge \mathrm{d}x^j - 2g_{ij} \mathrm{d}x^i \wedge \mathrm{d}y^j$ be the almost Finsler form associated with a homogeneous almost Finsler structure defined on a tangent bundle $T(M)$. Let $\sigma = \sigma(x, y)$ be a scalar field on $T(M)$ which is positively homogeneous of degree 0 for y^i. In order that $e^\sigma \Omega$ be closed, it is necessary and sufficient that $e^\sigma g_{ij}$ is a Finsler metric and the relation* (3.2) *holds good.*

Let g be a Finsler metric, Ω^* be the Finsler form associated with g, and $\sigma = \sigma(x)$ be a scalar field on M. Then $\tilde{g} = e^{\sigma(x)} g$ is a Finsler metric. So, let $\tilde{\Omega}^*$ be the Finsler form associated with $\tilde{g}$. Then we have

$$\tilde{\Omega}^* = e^{\sigma(x)} \Omega^* + e^{\sigma(x)} y^m(\partial_i \sigma g_{jm} - \partial_j \sigma g_{im}) .$$

Therefore, the condition $\tilde{\Omega}^* = e^{\sigma(x)} \Omega^*$ is written as $(\partial_i \sigma g_{jm} - \partial_j \sigma g_{im}) y^m = 0$. Applying the differentiation $\dot{\partial}_k$ and multiplying by g^{jk}, we have $\partial_i \sigma = 0$, i.e., σ is constant. Conversely, if σ is constant, it is evident that $\tilde{\Omega}^* = e^\sigma \Omega^*$. Thus we obtain

Theorem 3.3. *Let g and $\tilde{g}$ be two Finsler metrics defined on M, that are conformal to each other, namely, $\tilde{g} = e^{\sigma(x)} g$. Let Ω^* and $\tilde{\Omega}^*$ be the Finsler forms associated with g and $\tilde{g}$ respectively. Then $\tilde{\Omega}^* = e^{\sigma(x)} \Omega^*$ holds true if and only if $\tilde{g}$ is homothetic to g.*

4 - Hamilton vector fields in $T(M)$

Let V be a vector field in $T(M)$ and Q be the standard tangent structure tensor. With respect to a local canonical coordinate, V and Q are written as $V = v^i(x, y) \partial/\partial x^i + \bar{v}^i(x, y) \partial/\partial y^i$ and $Q = (Q^A_B) = \begin{pmatrix} 0 & 0 \\ \delta^i_j & 0 \end{pmatrix}$. Now, calculating the

Lie derivation $\mathcal{L}_V Q$, we have

$$\mathcal{L}_V Q^i_j = -\dot{\partial}_j v^i, \qquad \mathcal{L}_V Q^i_{\bar{j}} = 0, \qquad \mathcal{L}_V Q^{\bar{i}}_j = -\dot{\partial}_j v^{\bar{i}} + \partial_j v^i, \qquad \mathcal{L}_V Q^{\bar{i}}_{\bar{j}} = \dot{\partial}_j v^i .$$

Therefore, if $\mathcal{L}_V Q = F610$ holds, V must take the form

$$V = v^i(x)\,\partial/\partial x^i + (y^m \partial_m v^i(x) + u^i(x))\,\partial/\partial y^i .$$

And the converse is also true. Here, $v^i(x)\,\partial/\partial x^i + y^m \partial_m v^i(x)\,\partial/\partial y^i$ is called *the complete lift of a vector field* $v(x) = v^i(x)\,\partial/\partial x^i$ *to the tangent bundle* $T(M)$ and is denoted by $(v(x))^c$, and $u^i(x)\,\partial/\partial y^i$ is called *the vertical lift of a vector field* $u(x) = u^i(x)\,\partial/\partial x^i$ *to* $T(M)$ and is denoted by $(u(x))^v$ ([7], [20]). Hence we obtain

Theorem 4.1. *Let V be a vector field in a tangent bundle $T(M)$ and Q be the standard tangent structure tensor of $T(M)$. $\mathcal{L}_V Q = 0$ holds good if and only if $V = (v(x))^c + (u(x))^v$ where $(v(x))^c$ is the complete lift of a vector field $v(x)$ in M and $(u(x))^v$ is the vertical lift of a vector field $u(x)$ in M.*

Now, we suppose that the tangent bundle $T(M)$ admits a homogeneous almost Finsler structure $\mathcal{F}^*_1$. Let V be a vector field in $T(M)$. In what follows, we consider the case where the local 1-parameter group of local transformations generated by V preserves the structures $\mathcal{F}^*_1$. The condition to be demanded is written as $\mathcal{L}_V Q = 0$ and $\mathcal{L}_V \Omega = 0$. By virtue of Theorem 4.1, it is enough to consider the two cases where V is the complete lift or V is the vertical lift of a vector field in the base manifold M.

First, we consider the case where V is the complete lift of a vector field $v(x)$ in M. Now, let us calculate $\mathcal{L}_{v^c}\, \omega_{AB} = 0$ for $(\omega_{AB}) = \begin{pmatrix} \omega_{ij} & \omega_{i\bar{j}} \\ \omega_{\bar{i}j} & \omega_{\bar{i}\bar{j}} \end{pmatrix} = \begin{pmatrix} \alpha_{ij} & -g_{ij} \\ g_{ij} & 0 \end{pmatrix}$. Using the relations

$$\mathcal{L}_V\, \omega_{AB} = V^D \frac{\partial \omega_{AB}}{\partial x^D} + \frac{\partial V^D}{\partial x^a}\, \omega_{DB} + \omega_{AD} \frac{\partial V^D}{\partial x^B}, \qquad v^c = v^i(x)\,\partial/\partial x^i + y^m \partial_m v^i(x)\,\partial/\partial y^i$$

after some calculation, we get

$$\mathcal{L}_{v^c}\, \omega_{\bar{i}\bar{j}} = 0, \qquad \mathcal{L}_{v^c}\, \omega_{\bar{i}j} = v^h \partial_h g_{ij} + y^m \partial_m v^h \dot{\partial}_h g_{ij} + \partial_i v^h g_{hj} + g_{ih} \partial_j v^h = \mathcal{L}_v g_{ij}$$

$$\mathcal{L}_{v^c}\, \omega_{ij} = v^h \partial_h \alpha_{ij} + y^m \partial_m v^h \dot{\partial}_h \alpha_{ij} + \alpha_{hj} \partial_i v^h + \alpha_{ih} \partial_j v^h + g_{jh} \frac{\partial^2 v^h}{\partial x^i \partial x^m} y^m - g_{ih} \frac{\partial^2 v^h}{\partial x^j \partial x^m} y^m$$

where $\mathscr{L}_v g_{ij}$ is the well-known formula of the Lie derivative of the generalized metric g_{ij} [18].

As is well-known ([4], [8], [16], [17]), in a manifold admitting a symplectic structure whose associated 2-form is Ω, a vector field V satisfying $\mathscr{L}_V \Omega = 0$ is called a *Hamilton vector*. And similarly, in a manifold admitting an almost symplectic structure whose associated 2-form is Ω, a vector field V satisfying $\mathscr{L}_V \Omega = 0$ is said to be an *almost Hamilton vector*. Now we obtain

Theorem 4.2. *Let $T(M)$ be a tangent bundle admitting a homogeneous almost Finsler structure $\mathscr{F}_1^*$, let $\Omega = \alpha_{ij}\,dx^i \wedge dx^j - 2g_{ij}\,dx^i \wedge dy^j$ be the almost Finsler form associated with $\mathscr{F}_1^*$, and let $v = v^i(x)\,\partial/\partial x^i$ be a vector field in the base manifold M. Then, the complete lift of v is an almost Hamilton vector of $\mathscr{F}_1^*$ if and only if*

(1) *v is a Killing vector field of the generalized metric g_{ij}*

(2) $v^h \partial_h \alpha_{ij} + y^m \partial_m v^h \dot{\partial}_h \alpha_{ij} + \partial_i v^h \alpha_{hj} + \alpha_{ih} \partial_j v^h + g_{jh} \dfrac{\partial^2 v^h}{\partial x^i \partial x^m} y^m - g_{ih} \dfrac{\partial^2 v^h}{\partial x^j \partial x^m} y^m = 0$

hold good.

In the case where $d\Omega = 0$, i.e., g_{ij} is a Finsler metric and $\alpha_{ij} = y^m(\partial_i g_{jm} - \partial_j g_{im})$, the left hand side of the condition (2) of Theorem 4.2 can be rewritten, after some calculation due to $y^m \dot{\partial}_h g_{im} = 0$, as

$$v^h y^m \frac{\partial^2 g_{im}}{\partial x^h \partial x^i} - v^h y^m \frac{\partial^2 g_{im}}{\partial x^h \partial x^j} + y^m \partial_m v^h \partial_i g_{jh} - y^m \partial_m v^h \partial_j g_{ih} + y^m \partial_i v^h \partial_h g_{jm}$$

$$- y^m \partial_i v^h \partial_j g_{hm} + y^m \partial_j v^h \partial_i g_{hm} - y^m \partial_j v^h \partial_h g_{im} + y^m g_{jh} \frac{\partial^2 v^h}{\partial x^i \partial x^m} - y^m g_{ih} \frac{\partial^2 v^h}{\partial x^j \partial x^m}.$$

Thus we can rewrite the condition (2) as

$$\partial_i(y^m \mathscr{L}_v g_{jm}) - \partial_j(y^m \mathscr{L}_v g_{im}) = 0.$$

Therefore we obtain

Theorem 4.3. *Let g be a Finsler metric of a manifold M, $v = v^i(x)\,\partial/\partial x^i$ be a vector field in M and $\mathscr{F}_1^*$ be the symplectic structure on $T(M)$ derived from $\Omega^* = d(2y^m g_{mj}\,dx^j)$. Then v^c is a Hamilton vector of $\mathscr{F}_1^*$ if and only if v is a Killing vector of the Finsler metric g.*

It is well known ([4], [11], [16]) that, for any p-form, the relation $\mathcal{L}_V = \mathrm{i}_V \mathrm{d} + \mathrm{d}\mathrm{i}_V$ holds good where i_V is the *interior product* by V and d is the *exterior differential* operator. If v is a Killing vector field of a Finsler metric g, then we have $\mathcal{L}_{v^c}\Omega^* = 0$. Of course, $\mathrm{d}\Omega^* = 0$ holds. So, we have $\mathrm{d}\mathrm{i}_{v^c}\Omega^* = 0$. That is, the so-called Hamilton system $\mu = \omega_{BA}(v^c)^B \mathrm{d}x^A$ is closed. Putting $H_A = \omega_{AB}(v^c)^B$, we have

$$H_i = y^r\{(\partial_m g_{ir} - \partial_i g_{mr})\, v^m + g_{mi}\, \partial_r v^m\} \qquad H_{\bar{i}} = -\, g_{mi}\, v^m\,.$$

The equation $y^r \mathcal{L}_v g_{ir} = 0$ leads us to $H_i = -\, y^r v^m \partial_i g_{mr} - y^r \partial_i v^m g_{mr}$. Then we have $\mu = -\, \mathrm{d}(g_{mr} y^r v^m)$. That is, μ is an exact form and $H = g_{mr} y^r v^m$ is a Hamilton function of $\mathcal{F}_{\mathrm{I}}^*$ ([4], [8], [16], [17]). Thus we obtain

Theorem 4.4. *Suppose that a manifold M admits a Finsler metric g and a Killing vector field $v = v^i(x)\,\partial/\partial x^i$ of g. Concerning the symplectic structure $\mathcal{F}_{\mathrm{I}}^*$ derived from $\Omega^* = \mathrm{d}(2y^m g_{mj}\, \mathrm{d}x^j)$, $H = g_{mr} y^r v^m$ is the Hamilton function with respect to the Hamilton vector v^c in $T(M)$.*

In the case of Theorem 4.4, the so-called Hamilton equation is written as

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = v^i = g^{im}\frac{\partial H}{\partial y^m}, \qquad \frac{\mathrm{d}y^i}{\mathrm{d}t} = y^m\, \partial_m v^i = -\, g^{im}\frac{\partial H}{\partial x^m} + y^p(\partial_h g_{mp} - \partial_m g_{hp})\, g^{ih} g^{mr}\frac{\partial H}{\partial y^r}.$$

It is a matter of course that the Hamilton function is constant along the integral curve of the Hamilton vector v^c.

Next, we consider the case where $V = u^v$, u being a vector field on M. Calculating $\mathcal{L}_{u^v}\, \omega_{AB}$, we have

$$\mathcal{L}_{u^v}\ \omega_{ij} = u^m\, \dot{\partial}_m \alpha_{ij} + g_{mj}\, \partial_i u^m - g_{im}\, \partial_j u^m \qquad \mathcal{L}_{u^v}\ \omega_{i\bar{j}} = u^m\, \dot{\partial}_m g_{ij} \qquad \mathcal{L}_{u^v}\ \omega_{\bar{i}\bar{j}} = 0\,.$$

Thus we obtain

Theorem 4.5. *Let $T(M)$ be a tangent bundle admitting a homogeneous almost Finsler structure $\mathcal{F}_{\mathrm{I}}^*$, let $\Omega = \alpha_{ij}\, \mathrm{d}x^i \wedge \mathrm{d}x^j - 2g_{ij}\, \mathrm{d}x^i \wedge \mathrm{d}y^j$ be the almost Finsler form associated with $\mathcal{F}_{\mathrm{I}}^*$, and let $u = u^i(x)\,\partial/\partial x^i$ be a vector field in the base manifold M. Then, the vertical lift of u is an almost Hamilton vector of $\mathcal{F}_{\mathrm{I}}^*$*

if and only if

$$(1)\ u^m \dot{\partial}_m g_{ij} = 0 \qquad (2)\ u^m \dot{\partial}_m \alpha_{ij} + g_{mj}\partial_i u^m - g_{im}\partial_j u^m = 0$$

hold good.

Here we consider the case where $d\Omega = 0$, i.e., g is a Finsler metric and $\Omega = \Omega^*$. By virtue of (3.1) we have

$$u^m \dot{\partial}_m \alpha_{ij} + g_{mj}\partial_i u^m - g_{im}\partial_j u^m = u^m(\partial_i g_{jm} - \partial_j g_{im}) + g_{jm}\partial_i u^m - g_{im}\partial_j u^m .$$

Let $\overset{*}{\nabla}$ be the covariant differentiation with respect to the Cartan's Finsler connection $\overset{*}{\Gamma}{}^i_{jk}$ ([12], [18]). Using the condition $u^m \dot{\partial}_m g_{ij} = 0$, and the well-known relation $\overset{*}{\nabla}_k g_{ij} = 0$, we have

$$u^m(\partial_i g_{jm} - \partial_j g_{im}) + g_{ir}\overset{*}{\Gamma}{}^r_{mj} u^m - g_{jr}\overset{*}{\Gamma}{}^r_{mi} u^m = 0 .$$

Hence, we can rewrite the condition (2) as $\overset{*}{\nabla}_i(g_{jm}u^m) - \overset{*}{\nabla}_j(g_{im}u^m) = 0$. Therefore we obtain

Theorem 4.6. *Let g be a Finsler metric of a manifold M, let $u = u^i(x)\,\partial/\partial x^i$ be a vector field in M and let $\mathscr{F}^*_1$ be the homogeneous almost Finsler structure on $T(M)$ derived from $\Omega^* = d(2y^m g_{mj}\, dx^j)$. Then the vertical lift of u is a Hamilton vector of the symplectic structure $\mathscr{F}^*_1$ if and only if*

$$(1)\ u^m \dot{\partial}_m g_{ij} = 0 \qquad (2)\ \overset{*}{\nabla}_i(g_{jm}u^m) = \overset{*}{\nabla}_j(g_{im}u^m)$$

hold good where $\overset{*}{\nabla}$ means the covariant differentiation with respect to the Cartan's Finsler connection $\overset{*}{\Gamma}{}^i_{jk}$.

In the case of Theorem 4.6, the Hamilton system μ is written as $\mu = g_{im}u^m\, dx^i$. This μ is, naturally, a closed 1-form, however, is not always an exact form.

5 - $D(GL(n, R))$-structures

Now we assume that $T(M)$ admits a G-structure $\mathscr{B}^*$ which is depending on $\mathscr{F}^*_0$. If M is covered by a system of local coordinate neighbourhoods $\{(U, x^i)\}$ such that the natural frame $\{\partial/\partial x^A\}$ of the canonical coordinate neighbourhood

$(\pi^{-1}(U),\ x^A\}$ for each $(U,\ x^i)$ is adapted to the structure $\mathscr{B}^*$, then the G-structure $\mathscr{B}^*$, which is depending on $\mathscr{F}_0^*$, is called *integrable.*

Putting

$$D(GL(n,\ R)) = \{\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \mid A \in GL(n,\ R)\}$$

we see that $D(GL(n,\ R))$ is a Lie subgroup of $T(n)$. In this section, we treat the case where $T(M)$ admits a $D(GL(n,\ R))$-structure depending on $\mathscr{F}_0^*$, and denote it simply by $\mathscr{G}^*$.

If we put $P_0 = \begin{pmatrix} E_n & 0 \\ 0 & -E_n \end{pmatrix}$, then we see that $TP_0 = P_0 T$ holds for any $T \in D(GL(n,\ R))$. Hence, if $T(M)$ admits a structure $\mathscr{G}^*$ then it also admits an almost product structure. Let $\{Z_A\}$ be an adapted frame of $\mathscr{G}^*$ in each $(\pi^{-1}(U),\ x^A)$, and let us put $Z_A = \gamma^B{}_A \partial_B$, then $\Gamma = (\gamma^B{}_A)$ is written as

$$\Gamma = (\gamma^B{}_A) = \begin{pmatrix} \gamma^i{}_a & 0 \\ \sigma^i{}_a & \gamma^i{}_a \end{pmatrix}.$$

Putting $\gamma = (\gamma^i{}_a)$ and $\sigma = (\sigma^i{}_a)$, we see $\Gamma^{-1} = \begin{pmatrix} \gamma^{-1} & 0 \\ -\gamma^{-1}\sigma\gamma^{-1} & \gamma^{-1} \end{pmatrix}$. Now $P = \Gamma P_0 \Gamma^{-1} = \begin{pmatrix} E_n & 0 \\ 2\sigma\gamma^{-1} & -E_n \end{pmatrix}$ satisfies $P^2 = E_{2n}$ and becomes a globally defined (1.1)-tensor field on $T(M)$, i.e., *P is the almost product tensor field associated with the given almost product structure* [3].

Putting $N = (N^i{}_j) = -\sigma\gamma^{-1}$, we see, as is well-known, that *N is a non-linear connection defined on $T(M)$*. Of course, $N^i{}_j$ is $(1)p$-homogeneous for y. Now we show

Theorem 5.1. *A tangent bundle $T(M)$ admits a structure $\mathscr{G}^*$ (namely, a $D(GL(n,\ R))$-structure depending on $\mathscr{F}_0^*$), if and only if the underlying manifold M admits a non-linear connection.*

Proof. The necessity is shown already. So, we show the condition is sufficient. In each $(\pi^{-1}(U),\ x^A)$, let us put $X_i = \partial/\partial x^i - N^m{}_i \partial/\partial y^m$ and $X_{\bar{i}} = Y_i = \partial/\partial y^i$, then $\{X_A\}$ is a $2n$-frame in each $\pi^{-1}(U)$, which we call the *N-frame* hereafter. Let $\{\bar{X}_A\}$ be the N-frame in $(\pi^{-1}(\bar{U}),\ \bar{x}^A)$. If $U \cap \bar{U} \neq \phi$, it is easy to see that $\bar{Y}_m \dfrac{\partial \bar{x}^m}{\partial x^i} = Y_i$ and $\bar{X}_m \dfrac{\partial \bar{x}^m}{\partial x^i} = X_i$ in $\pi^{-1}(U) \cap \pi^{-1}(\bar{U})$. So, if we put

$X_A = T^B{}_A \bar{X}_B$, we have $(T^B{}_A) = \begin{pmatrix} \dfrac{\partial \bar{x}^i}{\partial x^j} & 0 \\ 0 & \dfrac{\partial \bar{x}^i}{\partial x^j} \end{pmatrix} \in D(GL(n, R))$. Since $N^i{}_j$ is (1)p-homogeneous with respect to y, we see that $T(M)$ admits a structure $\mathscr{G}^*$ and the N-frame is an adapted frame of the structure $\mathscr{G}^*$ in each $(\pi^{-1}(U), x^A)$.

Next, if we put $J_0 = (J^A{}_B) = \begin{pmatrix} 0 & -E_n \\ E_n & 0 \end{pmatrix}$, then we see directly that $TJ_0 = J_0 T$ holds for any $T \in D(GL(n, R))$. This means that $D(GL(n, R)) \subset GL(n, C)$, namely, $T(M)$ *admits an almost complex structure* if $T(M)$ admits a structure $\mathscr{G}^*$. The almost complex structure tensor F associated with this structure is given by $F = \Gamma J_0 \Gamma^{-1}$. Of course, F is a globally defined (1, 1)-tensor field on $T(M)$ satisfying $F^2 = -E_{2n}$ [3]. The components of F with respect to the canonical coordinate $\{x^A\}$ are given by ([5], [7], [9], [12], [19])

$$F = \Gamma J_0 \Gamma^{-1} = \begin{pmatrix} -N & -E_n \\ E_n + N^2 & N \end{pmatrix}.$$

Moreover, the components of F with respect to the N-frame $\{X_A\}$ are given by J_0 itself and also F satisfies $F(X_i) = Y_i$, $F(Y_i) = -X_i$. According to Matsumoto [12], this almost complex structure is called *the almost complex N-structure*. Now we show

Theorem 5.2. *In order that a tangent bundle $T(M)$ admits an integrable $D(GL(n, R))$-structure depending on $\mathscr{F}^*_0$, it is necessary and sufficient that the underlying manifold M is locally affine.*

Proof. If $T(M)$ admits the structure $\mathscr{G}^*$ which is integrable, then M is covered by a system of local coordinate neighborhoods $\{(U, x^i)\}$ such that the natural frame $\{\partial/\partial x^A\}$ of each $(\pi^{-1}(U), x^A)$ is adapted to the structure $\mathscr{G}^*$. On the other hand, the N-frame is also an adapted frame of the structure $\mathscr{G}^*$ in $(\pi^{-1}(U), x^A)$. So, we have $\partial/\partial x^A = T^B{}_A X_B$ where $(T^B{}_A) \in D(GL(n, R))$. That is, $\partial/\partial x^i = T^m{}_i\,(\partial/\partial x^m - N^r{}_m \partial/\partial y^r)$ and $\partial/\partial y^i = T^m{}_i \partial/\partial y^m$. These yield $T^m{}_i = \delta^m{}_i$ and $N^i{}_j = 0$. Next, let $\{\bar{U}, \bar{x}^i\}$ be another coordinate neighbourhood satisfying the above. If $U \cap \bar{U} \neq \phi$, then, in $\pi^{-1}(U) \cap \pi^{-1}(\bar{U})$, the relations

$$\bar{N}^i{}_r \frac{\partial \bar{x}^r}{\partial x^j} = \frac{\partial \bar{x}^i}{\partial x^r} N^r{}_j - \frac{\partial^2 \bar{x}^i}{\partial x^j \partial x^m} y^m \qquad N^i{}_j = 0 \qquad \bar{N}^i{}_j = 0$$

hold, and these lead us to $\dfrac{\partial^2 \bar{x}^i}{\partial x^j \partial x^k} = 0$, that is, M is locally affine.

Conversely, if M is locally affine, there globally exists a flat affine connection $\Gamma^i_{jk}(x)$ on M. Then $T(M)$ is endowed with a non-linear connection such as $N^i{}_j = \Gamma^i_{mj}(x)\, y^m$. Owing to Theorem 5.1, $T(M)$, therefore, admits a $D(GL(n, R))$-structure depending on $\mathscr{F}^*_0$, i.e., a structure $\mathscr{C}^*$, whose N-frame is adapted to $\mathscr{C}^*$. Since $\Gamma^i_{jk}(x)$ is a global flat affine connection on M, M is covered by a system of local coordinate neighbourhoods $\{(U, x^i)\}$ such that $\Gamma^i_{jk}(x) = 0$ holds in each U. Then, in each U, $N^i{}_j = 0$ holds, from which we have $X_i = \partial/\partial x^i$ and $Y_i = \partial/\partial y^i$. Namely, the canonical natural frame $\{\partial/\partial x^A\}$ is adapted to $\mathscr{C}^*$, in each $(\pi^{-1}(U), x^A)$. Consequently, the proof is complete.

6 - $D(O(n))$-structures

In this section we consider the *Lie group* defined by

$$D(O(n)) = \{\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} | A \in O(n)\}.$$

It is obvious that $T(O(n)) \cap F(n) = D(O(n))$ and the Lie algebra of $D(O(n))$ is given by $\mathfrak{d}(\mathfrak{o}(n)) = \{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} | a \in \mathfrak{o}(n)\}$. Now we consider the case where $T(M)$ admits a homogeneous $D(O(n))$-structure depending on $\mathscr{F}^*_0$, which we denote by $\mathscr{F}^*_2$.

In a canonical coordinate neighbourhood $\{\pi^{-1}(U), x^A\}$, let $\{Z_A\}$ be a $2n$-frame adapted to the structure $\mathscr{F}^*_2$ and represent $Z_A = \gamma^B{}_A\, \partial_B$. Since $\Gamma = (\gamma^B{}_A) = \begin{pmatrix} \gamma^i{}_a & 0 \\ \sigma^i{}_a & \gamma^i{}_a \end{pmatrix} \in T(n)$, $\det|\gamma^i{}_a| \neq 0$ and $\gamma^i{}_a$ are positively homogeneous of degree 0 with respect to y^i and $\sigma^i{}_a$ are positively homogeneous of degree 1 with respect to y^i.

First, it is clear that $D(O(n)) \subset D(GL(n, R))$. So, if $T(M)$ admits a structure $\mathscr{F}^*_2$, then it admits a structure $\mathscr{C}^*$, that is, $T(M)$ admits a non-linear connection N, given by $N = -\sigma\gamma^{-1}$.

On the other hand, it is clear that $D(O(n)) \subset O(2n)$. So, if $T(M)$ admits the structure $\mathscr{F}^*_2$, it also admits a Riemann metric. That is, if we define an inner product in each $\pi^{-1}(U)$ by $\langle Z_A, Z_B \rangle = \delta_{AB}$, the inner product in each $\pi^{-1}(U)$ gives $T(M)$ globally a positive definite Riemann metric G. The components of G with respect to the frame $\{X_A\}$ are written as

$$\langle X_i,\ X_j\rangle = \sum_{a=1}^{n} \beta^a{}_i\beta^a{}_j = g_{ij} \qquad \langle X_i,\ Y_j\rangle = 0 \qquad \langle Y_i,\ Y_j\rangle = g_{ij}$$

where $(\beta^a{}_i) = \beta = \gamma^{-1}$.

The functions g_{ij} are positively homogeneous of degree 0 with respect to y^i and they give M a so-called generalized metric.

Conversely, if a manifold M admits a generalized metric g_{ij} and a non-linear connection $N^i{}_j$, a proof similar to that of Theorem 2.4 shows us that $T(M)$ admits a homogeneous $D(O(n))$-structure depending on $\mathscr{F}_0^*$. Therefore we obtain **[9]**

Theorem 6.1. *A necessary and sufficient condition for a tangent bundle $T(M)$ to admit a structure $\mathscr{F}_2^*$* (i.e., *a homogeneous $D(O(n))$-structure depending on $\mathscr{F}_0^*$) is that the manifold M admits a generalized metric and a non-linear connection.*

If $T(M)$ admits the structure $\mathscr{F}_2^*$, because of the fact $D(O(n)) \subset F(n)$, $T(M)$ admits an *almost symplectic structure.* Moreover we can see that the functions g_{ij} in the 2-form $\Omega = \alpha_{ij}\,dx^i \wedge dx^j - 2g_{ij}\,dx^i \wedge dy^j$ coincide with the components g_{ij} of the given generalized metric. The functions α_{ij} in Ω are given by $\alpha_{ij} = \sum_a (\beta^{\bar{a}}{}_i\beta^a{}_j - \beta^a{}_i\beta^{\bar{a}}{}_j)$, and the relations $(\beta^{\bar{a}}{}_j) = -\beta\sigma\beta$ and $-\sigma\beta = N$ hold. Hence, $\beta^{\bar{a}}{}_j = \beta^a{}_m N^m{}_j$ and $\alpha_{ij} = g_{jr}N^r{}_i - g_{ir}N^r{}_j$. With respect to these quantities, we can easily verify

$$d\Omega = 2[\partial_k(g_{jr}N^r{}_i)\,dx^k \wedge dx^i \wedge dx^j + \dot{\partial}_k(g_{jr}N^r{}_i)\,dy^k \wedge dx^i \wedge dx^j$$
$$+ \partial_j g_{ik}\,dy^k \wedge dx^i \wedge dx^j - \dot{\partial}_k g_{ij}\,dy^k \wedge dx^i \wedge dy^j].$$

From this relation, as well as from Theorem 3.1, we can prove **[9]**.

Theorem 6.2. *If a tangent bundle $T(M)$ admits a structure $\mathscr{F}_2^*$* (i.e., *a homogeneous $D(O(n))$-structure depending on $\mathscr{F}_0^*$), $T(M)$ admits an almost symplectic form $\Omega = -\{(g_{ir}N^r{}_j - g_{jr}N^r{}_i)\,dx^i \wedge dx^j + 2g_{ij}\,dx^i \wedge dy^j\}$. Moreover Ω is closed if and only if g_{ij} is a Finsler metric on M and Ω coincides with the Finsler form Ω^* associated with the Finsler metric.*

Moreover we can prove

Theorem 6.3. *In order that a tangent bundle $T(M)$ admits an integrable $D(O(n))$-structure depending on $\mathscr{F}_0^*$, it is necessary and sufficient that the underlying manifold M admits a flat Riemann metric.*

Proof. First, let us consider a tangent bundle $T(M)$ admitting an integrable $\mathscr{F}_2^*$ structure. Due to the definition, $T(M)$ is covered by a system of canonical coordinate neighbourhoods $\{(\pi^{-1}(U),\ x^A)\}$ such that the natural frame $\{\partial/\partial x^A\}$ is adapted to the structure $\mathscr{F}_2^*$. Then $\langle \partial/\partial x^A,\ \partial/\partial x^B\rangle = \delta_{AB}$ holds. That is, $g_{ij}=\delta_{ij}$ holds true with respect to each $(U,\ x^i)$. Of course, $\{(U,\ x^i)\}$ covers M. Hence M is locally Euclidean.

Conversely, if M admits a flat Riemann metric g_{ij}, then M is covered by a system of local coordinate neighbourhoods $\{(U,\ X^i)\}$ with respect to which $g_{ij}=\delta_{ij}$ holds always. Then $\left\{{i \atop jk}\right\}=0$ holds. Now, the system of the canonical coordinate neighbourhoods $\{(\pi^{-1}(U),\ x^A)\}$ covers $T(M)$. With respect to these coordinate neighbourhoods, the non-linear connection $N^i{}_j=\left\{{i \atop mj}\right\}y^m$ vanishes. Hence the N-frame $\{X_i,\ Y_i\}$ for the non-linear connection coincides with the canonical natural frame $\{\partial/\partial x^A\}$. On the other hand, according to Theorem 6.1, g_{ij} and $N^i{}_j$ determine a structure $\mathscr{F}_2^*$ in $T(M)$ and the relations $\langle X_i,\ X_j\rangle = g_{ij}$, $\langle Y_i,\ Y_j\rangle = g_{ij}$ and $\langle X_i,\ Y_j\rangle = 0$ hold. Hence we have $\langle \partial/\partial x^A,\ \partial/\partial x^B\rangle = \delta_{AB}$. Thus the natural frame $\{\partial/\partial x^A\}$ is adapted to the structure $\mathscr{F}_2^*$. That is, the structure $\mathscr{F}_2^*$ is integrable.

Next, we denote by $\overset{N}{\nabla}$ the h-covariant derivative with respect to the non-linear connection N, that is, for any quasi tensor field T, for example, of $(1,\ 1)$-type,

$$\overset{N}{\nabla}_k T^i{}_j = \partial_k T^i{}_j - \dot{\partial}_m T^i{}_j N^m{}_k + T^m{}_j \dot{\partial}_m N^i{}_k - T^i{}_m \dot{\partial}_j N^m{}_k \,.$$

If N satisfies $\overset{N}{\nabla}_k g_{ij}=0$, then the non-linear connection N is said to be *metrical.* Now we show

Theorem 6.4. *In order that a manifold M is a locally Minkowski manifold, it is necessary and sufficient that the tangent bundle $T(M)$ admits a structure $\mathscr{F}_2^*$ satisfying*

(1) *The structure $\mathscr{O}^*$ induced from $\mathscr{F}_2^*$ is integrable*

(2) *The almost symplectic 2-form Ω induced from $\mathscr{F}_2^*$ is closed*

(3) *The non-linear connection derived from $\mathscr{G}^*$ is metrical with respect to the generalized metric derived from $\mathscr{F}_2^*$.*

Proof. Let M be locally Minkowskian and g_{ij} be the metric tensor. Then M is covered by a system local coordinate neighbourhood $\{(U, x^i)\}$ such that $\partial_k g_{ij}=0$ holds good in each U. Then, in these coordinate neighbourhoods, the Cartan's Finsler connection $\overset{*}{\Gamma}{}^i_{jk}$ and the Cartan's non-linear connection $G^i{}_j$ vanish. Now, g_{ij} and $G^i{}_j$ induce a structure $\mathscr{F}_2^*$ in $T(M)$. We consider this structure $\mathscr{F}_2^*$. The N-frame associated with $G^i{}_j$ is an adapted frame of the structure $\mathscr{G}^*$ determined by $G^i{}_j$, and $G^i{}_j=0$ holds in each $\pi^{-1}(U)$. So, the natural frame $\{\partial/\partial x^A\}$ is adapted to the structure $\mathscr{G}^*$. Of course, by definition, $\mathscr{G}^*$ coincides with the $D(GL(n, R))$-structure induced from $\mathscr{F}_2^*$. Namely, the structure $\mathscr{G}^*$ satisfies the condition (1). On the other hand, it is obvious in each U that

$$\overset{G}{\nabla}_k g_{ij} = \partial_k g_{ij} - \dot{\partial}_m g_{ij} G^m{}_k - g_{mj}\dot{\partial}_i G^m{}_k - g_{im}\dot{\partial}_j G^m{}_k = 0 .$$

Hence the condition (3) is satisfied. And, of course, g_{ij} is a Finsler metric. Hence, owing to Theorem 6.2 the structure $\mathscr{F}_2^*$ satisfies the condition (2).

Conversely, let us assume that $T(M)$ admits a structure $\mathscr{F}_2^*$ satisfying (1), (2) and (3). Then there exists a non-linear connection N and a generalized metric g. Now, according to Theorem 6.2, the condition (2) tells us that g is a Finsler metric. And the condition (1) implies that $T(M)$ is covered by a system of local coordinate neighbourhoods $\{(\pi^{-1}(U), x^A)\}$ such that $\{(U, x^i)\}$ covers M and $N^i{}_j=0$ holds in each $\pi^{-1}(U)$. On the other hand, the condition (3) means $\overset{N}{\nabla} g=0$. Then, with respect to these coordinates, $\partial_k g_{ij}=0$ holds. Namely, M is covered by a system of local coordinate neighbourhoods $\{(U, x^i)\}$ such that $\partial_k g_{ij}=0$ holds good in each U. Therefore M is locally Minkowskian.

7 - G-connections relative to the structure $\mathscr{F}_2^*$

First, we assume that a tangent bundle $T(M)$ admits a structure $\mathscr{F}_2$, that is, a $D(O(n))$-structure depending on $\mathscr{F}_0$, and $\{Z_A\}$ is the $2n$-frame adapted to $\mathscr{F}_2$. Now we shall treat G-connections relative to the structure $\mathscr{F}_2$. Let ∇ be such a connection. Then ∇ is a linear connection on $T(M)$ and satisfies $\nabla_U Z_A = \overset{\approx}{\Gamma}{}^B_{AC} U^C Z_B$ where $\overset{\approx}{\Gamma}{}^B_{AC} U^C \in \mathfrak{d}(\mathfrak{o}(n))$ for any $U = U^C Z_C$. The condition can be rewritten as

$$(*) \qquad \overset{\approx}{\Gamma}{}^{i}_{\bar{j}A}=0, \qquad \overset{\approx}{\Gamma}{}^{\bar{i}}_{jA}=0 \qquad \overset{\approx}{\Gamma}{}^{i}_{jA}=\overset{\approx}{\Gamma}{}^{\bar{i}}_{\bar{j}A}=-\overset{\approx}{\Gamma}{}^{j}_{iA}=-\overset{\approx}{\Gamma}{}^{\bar{j}}_{\bar{i}A}.$$

Let G be the Riemann metric defined on $T(M)$ derived from the structure $\mathscr{F}_2$. We have

$$(\nabla_U G)_{AB} = U(\langle Z_A,\ Z_B\rangle) - \langle \nabla_U Z_A,\ Z_B\rangle - \langle Z_A,\ \nabla_U Z_B\rangle$$

$$= -\langle \overset{\approx}{\Gamma}{}^{P}_{AQ} U^Q Z_P,\ Z_B\rangle - \langle Z_A,\ \overset{\approx}{\Gamma}{}^{P}_{BQ} U^Q Z_P\rangle = -\overset{\approx}{\Gamma}{}^{B}_{AQ} U^Q - \overset{\approx}{\Gamma}{}^{A}_{BQ} U^Q = 0.$$

Thus a G-connection ∇ relative to the structure $\mathscr{F}_2$ is a metrical connection for the Riemann metric derived from $\mathscr{F}_2$, and, from the above condition (*), the connection ∇ satisfies

$$(1) \quad \nabla_{Z_j} Z_i = \overset{\approx}{\Gamma}{}^{m}_{ij} Z_m \qquad (2) \quad \nabla_{Z_{\bar{j}}} Z_i = \overset{\approx}{\Gamma}{}^{m}_{i\bar{j}} Z_m$$

$$(3) \quad \nabla_{Z_j} Z_i = \overset{\approx}{\Gamma}{}^{m}_{ij} Z_{\bar{m}} \qquad (4) \quad \nabla_{Z_{\bar{j}}} Z_{\bar{i}} = \overset{\approx}{\Gamma}{}^{m}_{i\bar{j}} Z_{\bar{m}}.$$

Conversely, if the above conditions (1) (2) (3) (4) and $\nabla G = 0$ are all satisfied, the condition (*) is fulfilled. Consequently we obtain

Theorem 7.1. *In order that a linear connection ∇ on a tangent bundle admitting a structure $\mathscr{F}_2$* (i.e., *a $D(O(n))$-structure depending on $\mathscr{F}_0$*) *be a G-connection relative to the structure $\mathscr{F}_2$, it is necessary and sufficient that the conditions* (1), (2), (3), (4) *and $\nabla G = 0$ are satisfied, where G is the Riemann metric derived from the structure $\mathscr{F}_2$.*

Now, we add the homogeneity condition to the structure $\mathscr{F}_2$, that is, we consider the structure $\mathscr{F}_2^*$. Then, there exist a non-linear connection N and a generalized metric g on the manifold M. In addition, ∇ is a G-connection relative to the structure $\mathscr{F}_2^*$ if and only if, for any positively homogeneous vector U of degree 0 with respect to y^i, $\nabla_U Z_A = \overset{\approx}{\Gamma}{}^{B}_{AC} U^C Z_B$ where

$$(\overset{\approx}{\Gamma}{}^{B}_{AC} U^C) \in \{(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}) \ |a + {}^t a = 0,\ a \text{ is (0) } p\text{-phomogeneous with respect to } y^i\}$$

are fulfilled. Since $Z_a = \gamma^m{}_a X_m$ and $Z_{\bar{a}} = \gamma^m{}_a Y_m$ ($\gamma^m{}_a$ are positively homogeneous of degree 0 with respect to y^i), $\overset{\approx}{\Gamma}{}^{i}_{jk}$ is positively homogeneous of degree 0 for y^i

and $\tilde{\tilde{\Gamma}}^i_{jk}$ is positively homogeneous of degree -1 for y^i. Now, let us denote the components of ∇ with respect to $\{X_A\}$ by $\tilde{\Gamma}^A_{BC}$. Checking the conditions (1), (2), (3), (4) and $\nabla G=0$ in Theorem 7.1 with respect to the frame $\{X_A\}$, we have

Theorem 7.2. *On a tangent bundle admitting a structure $\mathscr{F}^*_2$ (i.e., a homogeneous $D(O(n))$-structure depending on $\mathscr{F}^*_0$), a linear connection ∇ is a G-connection relative to the structure $\mathscr{F}^*_2$ if and only if*

$$\nabla_{X_k}X_j = F^m_{jk}X_m \qquad \nabla_{X_k}Y_j = F^m_{jk}Y_m \qquad \nabla_{Y_k}X_j = C^m_{jk}X_m \qquad \nabla_{Y_k}Y_j = C^m_{jk}Y_m$$

hold where F^m_{jk} are positively homogeneous of degree 0 with respect to y^i and C^m_{jk} are positively homogeneous of degree -1 with respect to y^i, and

$$\partial_k g_{ij} - \dot{\partial}_m g_{ij} N^m{}_k - g_{im}F^m_{jk} - g_{jm}F^m_{ik} = 0 \qquad \dot{\partial}_k g_{ij} - g_{im}C^m_{jk} - g_{jm}C^m_{ik} = 0$$

hold good.

If a linear connection ∇ satisfies the first four conditions in Theorem 7.2, ∇ is called, as is well-known, *a linear connection of Finsler type* [12]. And the last two conditions are the condition for a Finsler connection (F, N, C) to be a *metrical connection* with respect to a generalized metric g. Consequently we can rewrite Theorem 7.2 as

Theorem 7.3. *On a tangent bundle admitting a structure $\mathscr{F}^*_2$, a linear connection ∇ is a G-connection relative to the structure $\mathscr{F}^*_2$ if and only if ∇ is a linear connection of Finsler type and, at the same time, the Finsler connection derived from ∇ and from the non-linear connection N is a metrical connection of the generalized metric.*

It follows from the definition that the components of the tensors ω, Q and P with respect to the frame $\{X_A\}$ have the following forms:

$$(\tilde{\omega}_{AB}) = \begin{pmatrix} 0 & -g \\ g & 0 \end{pmatrix} \qquad (\tilde{Q}^A{}_B) = \begin{pmatrix} 0 & 0 \\ E & 0 \end{pmatrix} \qquad (\tilde{P}^A{}_B) = \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}.$$

Let ∇ be a linear connection on $T(M)$. Then, by calculating the components of ∇Q, ∇F and $\nabla\omega$, we obtain

Theorem 7.4. *Let $T(M)$ be a tangent bundle admitting a structure $\mathscr{F}^*_2$. A* *linear connection* ∇ *on* $T(M)$ *is a linear connection of Finsler type if and only if* $\nabla Q=0$ *and* $\nabla P=0$ *hold good. Moreover the linear connection* ∇ *is a* G-*connection relative to the structure* $\mathscr{F}_2^*$ *if and only if* $\nabla Q=0$, $\nabla P=0$ *and* $\nabla\omega=0$ *hold good.*

The curvature and torsion with respect to a linear connection of Finsler type are shown by Matsumoto in [12]. However, we shall consider them from our viewpoint using our notation.

The torsion of a linear connection is given by

$$T(U,\ V)=\nabla_U V-\nabla_V U-[U,\ V].$$

For the components of T of a G-connection relative to the structure $\mathscr{F}_2^*$ with respect to the frame $\{X_i,\ Y_i\}$ (i.e., $T(X_B,\ X_C)=\tilde{T}^A_{BC}X_A$), we have

$$\tilde{T}^h_{jk}=-F^h_{jk}+F^h_{kj}\qquad \tilde{T}^h_{j\bar{k}}=-C^h_{jk}\qquad \tilde{T}^h_{\bar{j}k}=C^h_{kj}\qquad \tilde{T}^h_{\bar{j}\bar{k}}=0$$

$$\tilde{T}^{\bar{h}}_{jk}=-R^h_{jk}\qquad \tilde{T}^{\bar{h}}_{j\bar{k}}=F^h_{kj}-\dot{\partial}_k N^h{}_j\qquad \tilde{T}^{\bar{h}}_{\bar{j}k}=\dot{\partial}_j N^h{}_k-F^h_{jk},\qquad \tilde{T}^{\bar{h}}_{\bar{j}\bar{k}}=-C^h_{jk}+C^h_{kj}$$

where we put $R_j{}^h{}_k=-\partial_j N^h{}_k+\partial_k N^h{}_j+\dot{\partial}_r N^r{}_j-\dot{\partial}_r N^h{}_j N^r{}_k$ [12].

The curvature of a linear connection is given by

$$R(W;\ U,\ V)=(\nabla_U\nabla_V-\nabla_V\nabla_U-\nabla_{[U,V]})W.$$

For the components of R of a G-connection relative to the structure $\mathscr{F}_2^*$ with respect to the frame $\{X_i,\ Y_i\}$ (i.e., $R(X_B;\ X_C,\ X_D)=\tilde{R}^A{}_{BCD}X_A$), we have

$$\tilde{R}^i{}_{hjk}=-R_h{}^i{}_{jk}\qquad \tilde{R}^i{}_{hj\bar{k}}=-P_h{}^i{}_{jk}\qquad \tilde{R}^i{}_{h\bar{j}k}=P_h{}^i{}_{jk}\qquad \tilde{R}^i{}_{h\bar{j}\bar{k}}=-S_h{}^i{}_{jk}$$

$$\tilde{R}^{\bar{i}}{}_{\bar{h}jk}=-R_h{}^i{}_{jk}\qquad \tilde{R}^{\bar{i}}{}_{\bar{h}j\bar{k}}=-P_h{}^i{}_{jk}\qquad \tilde{R}^{\bar{i}}{}_{\bar{h}\bar{j}k}=P_h{}^i{}_{jk}\qquad \tilde{R}^{\bar{i}}{}_{\bar{h}\bar{j}\bar{k}}=-S_h{}^i{}_{jk}$$

$$\tilde{R}^{\bar{i}}{}_{hjk}=\tilde{R}^{\bar{i}}{}_{hj\bar{k}}=\tilde{R}^{\bar{i}}{}_{h\bar{j}k}=\tilde{R}^{\bar{i}}{}_{h\bar{j}\bar{k}}=0\qquad \tilde{R}^i{}_{\bar{h}jk}=\tilde{R}^i{}_{\bar{h}j\bar{k}}=\tilde{R}^i{}_{\bar{h}\bar{j}k}=\tilde{R}^i{}_{\bar{h}\bar{j}\bar{k}}=0$$

where we put [12]

$$R_h{}^i{}_{jk}=\partial_k F^i_{hj}-\dot{\partial}_m F^i_{hj}N^m{}_k-\partial_j F^i{}_{hk}+\dot{\partial}_m F^i_{hk}N^m{}_j+F^m_{hj}F^i{}_{mk}-F^m_{hk}F^i_{mj}+C^i_{hm}R^m_{jk}$$

$$P_h{}^i{}_{jk}=-(\partial_j C^i_{hk}-\dot{\partial}_m C^i_{hk}N^m{}_j+C^m_{hk}F^i_{mj}-C^i_{mk}F^m_{hj}-C^i_{hm}F^m_{kj})+C^i_{hm}(\dot{\partial}_k N^m{}_j-F^m_{kj})$$
$$+\dot{\partial}_k F^i_{hj}$$

$$S_h{}^i{}_{jk}=\dot{\partial}_k C^i_{hj}-\dot{\partial}_j C^i_{hk}+C^i_{mk}C^m_{hj}-C^i_{mj}C^m_{hk}.$$

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