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On the rationality of a certain class of cubic complexes (**)

1 - Introduction

Let V be the intersection of a smooth quadric hypersurface Q and a smooth cubic hypersurface X in $\mathbb{P}^5(C)$. V is a well known non rational Fano variety, though unirational (see $[7]_1$).

If we identify Q with the Grassmannian $G(1, 3)$ of lines of $\mathbb{P}^3(C)$, V is classically called a *cubic complex*.

Let V_n be the complete intersection of smooth Q and X containing $n \geqslant 1$ planes two by two meeting at one pont only. Conte proved that V_1 is not rational, (see $[4]_2$); he used Beauville's theory of conic bundles.

E. Ambrogio and D. Romagnoli proved the non rationality of V_2 and V_3 (see [2]). When $n \geqslant 4$ V_n is rational; it follows from the existence of a birational map between $G(1, 3)$ and $\mathbb{P}^4(C)$, under which some cubic complexes in $\mathbb{P}^5(C)$ correspond to cubic hypersurfaces in $\mathbb{P}^4(C)$ (see **3**; the idea is due to Fano, see [5]).

In this paper we study a conic bundle structure arising from V_n: it provides an example for which Beauville's theory fails. We prove the rationality of V_n, $n \geqslant 4$, as an application of some recent results given by Sarkisov and Iskovskih about the rationality of conic bundles (see [10], $[7]_{2,3,4}$).

These conic bundle structures also arise from cubic threefolds of $\mathbb{P}^4(C)$, so that our results work in this case too; we have outlined these further applications in Remark 5.4. In **6** we prove that V_7 always contains another plane, so that the 8 planes contained in V_8 are not in general position.

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We also prove $n \leq 8$: namely if $n \geq 9$ X splits into Q and into a hyperplane. As a consequence of our results we obtain a confirm to the Conjecture 8.3 of Iskovskih on the rationality of conic bundles (see $[7]_3$). The same techniques allow us to show that V_2 is not rational, according to [2], but in another way.

In a separated paper, [1], we also solved the problem of rationality for cubic complexes containing n planes with the remaining incidence conditions.

We wish to thank prof. A. Conte, who called this problem to our attention, and prof. F. Bardelli for many helpful conversations.

2 - Notations and preliminaries

Variety: by this term we mean an algebraic projective variety on $\boldsymbol{C}$.

$\mathbb{P}^r$: r-dimensional projective space on $\boldsymbol{C}$.

V_n: the complete intersection of a smooth quadric hypersurface and a smooth cubic hypersurface in $\mathbb{P}^5$, containing n planes two by two meeting at one point only.

P^s: s^{th} plane contained in V_n.

$Prym(\tilde{C}, C)$: *Prym* variety associated to the double covering $\tilde{C}$ of the curve C.

$J(Y)$: intermediate Jacobian of the 3-variety Y.

$H^*(Y, Z)$: cohomological ring with integer coefficients of the variety Y.

Def. 2.1. We call *conic bundle* a non singular variety V with a surjective morphism $h: V \to S$, where S is a smooth surface, satisfying the following condition: for every point s of S, the fibre $h^{-1}(s)$ is isomorphic to a conic, singular or not.

Def. 2.2. The conic bundle V is called *standard* if for every irreducible divisor D of S, $h^{-1}(D)$ is an irreducible divisor of V.

Def. 2.3. (See [10]). A triple (V, S, h), where $h: V \to S$ is a rational map whose generic fibre is an irreducible rational curve and S is a non singular surface, is called a *conic fibration* (c.f.) over S.

Remark 2.4. We use a term different from Sarkisov's to avoid confusion with 2.1.

Def. 2.5. A c.f. is called *regular* if h is a flat morphism of nonsingular varieties.

Remark 2.6. A regular c.f. such that every fibre is a conic, is a conic bundle according to Def. 2.1.

3 - Rationality of V_n when $n \geq 4$

In P^4 we choose $(z_1:z_2:z_3:z_4:z_5)$ as coordinates; the advantage of this unusual choice will be clear in the sequel (see Remark 5.4). We fix three lines in general position:

$$l_1:\ z_3 = z_4 = z_5 = 0$$

$$l_2:\ z_1 - z_3 = z_2 = z_5 = 0$$

$$l_3:\ z_1 = z_2 = z_4 = 0\,.$$

In P^5 we choose $(x_0:x_1:x_2:x_3:x_4:x_5)$ as coordinates. We consider the rational map $\Phi: P^4 \to P^5$ associated to $|0_{P^4}(2) - l_1 - l_2 - l_3|$.

It is easy to see that Φ is birational between P^4 and the hyperquadric Q of P^5 whose equation is

$$x_0 x_5 - x_1 x_4 + x_2 x_3 = 0\,. \tag{3.1}$$

We identify the smooth hyperquadric Q with $G(1,\ 3)$, the Grassmannian of lines of P^3.

Φ^{-1} is the restriction to Q of the rational map, from P^5 to P^4, associated to $|0_{P^5}(2) - \pi_{12} - \pi_{13} - \pi_{23}|$; where

$$\pi_{12}:\ x_1 = x_2 = x_5 = 0$$

$$\pi_{13}:\ x_0 = x_2 = x_4 = 0$$

$$\pi_{23}:\ x_3 = x_4 = x_5 = 0$$

are the images of the hyperplanes of P^4 spanned respectively by l_1, l_2; l_1, l_3; l_2, l_3. π_{12}, π_{13}, π_{23} meet two by two at one point only.

Under Φ, cubic threefolds in $\mathbb{P}^4$ containing l_1, l_2, l_3 correspond to the cubic complexes containing π_{12}, π_{13}, π_{23}. If we choose another plane ζ, meeting π_{12}, π_{13}, π_{23} at one point only, $\Phi^{-1}(\zeta)$ is a *plane* in $\mathbb{P}^4$; so that a cubic complex as V_n, $n \geqslant 4$, containing 4 (or more) such planes is birational to a cubic threefold containing a plane, which is singular and therefore rational. While a cubic complex containing 3 such planes, as V_3, is birational to a smooth cubic threefold, which is not rational (see $[4]_1$).

4 - Conic bundle structure arising from V_1

From now on V_n will be the complete intersection of the hyperquadric Q previously considered and a smooth cubic hypersurface X in $\mathbb{P}^5$.

We consider $n = 1$; we have V_1 containing only one plane P^1. Now we want to prove that a generic V_1 is singular and has 7 ordinary double points on P^1. Meanwhile we fix a coordinate system useful in the sequel.

Obviously a generic V_1 is smooth out of P^1; we determine the singular points on P^1. Every line in $\mathbb{P}^3$ is determined by a couple of points $(a_0 : a_1 : a_2 : a_3)$ and $(b_0 : b_1 : b_2 : b_3)$; then the point of $G(1, 3)$, corresponding to the line joining $(a_0 : a_1 : a_2 : a_3)$ and $(b_0 : b_1 : b_2 : b_3)$, has coordinates (see [6]):

$$
\begin{aligned}
&x_0 = a_0 b_1 - b_0 a_1 \qquad x_1 = a_0 b_2 - b_0 a_2 \\
(4.1) \qquad &x_2 = a_0 b_3 - b_0 a_3 \qquad x_3 = a_1 b_2 - b_1 a_2 \\
&x_4 = a_1 b_3 - b_1 a_3 \qquad x_5 = a_2 b_3 - b_2 a_3 .
\end{aligned}
$$

As the planes in V_n meet at one point only, we can suppose that they belong to only one ruling of Q; the ruling corresponding to the stars of lines in $\mathbb{P}^3$. We choose a coordinate system in $\mathbb{P}^3$ such that P^1 corresponds to the star of lines centered in $(1:0:0:0)$; by (4.1) P^1 has equations: $x_3 = x_4 = x_5 = 0$. So we can say that the generic X containing P^1 has equation

$$
\begin{aligned}
(4.2) \qquad &x_0^2 E_1 + x_1^2 F_1 + x_2^2 G_1 + x_0 x_1 H_1 + x_0 x_2 L_1 \\
&+ x_1 x_2 M_1 + x_0 N_2 + x_1 P_2 + x_2 Q_2 + R_3 = 0
\end{aligned}
$$

where

$$E_1 = E_1(x_3 : x_4 : x_5) = e_1 x_3 + e_2 x_4 + e_3 x_5 \qquad F_1 = F_1(x_3 : x_4 : x_5) = f_1 x_3 + f_2 x_4 + f_3 x_5$$

etc. are degree one homogeneous polynomials in x_3, x_4, x_5; N_2, P_2, Q_2 are of degree two; R_3 is of degree three.

A point on P^1 is singular for V_1 if and only if the hyperplanes tangent to Q and to X are the same. We confuse the letters Q and X, respectively, with the equations (3.1) and (4.2); then the partial derivatives of Q and X, evaluated at points of P^1, are:

$$Q_{x_0} = 0 \qquad Q_{x_1} = 0 \qquad Q_{x_2} = 0$$

$$Q_{x_3} = x_2 \qquad Q_{x_4} = -x_1 \qquad Q_{x_5} = x_0$$

$$X_{x_0} = 0 \qquad X_{x_1} = 0 \qquad X_{x_2} = 0$$

$$X_{x_3} = e_1 x_0^2 + f_1 x_1^2 + g_1 x_2^2 + h_1 x_0 x_1 + l_1 x_0 x_2 + m_1 x_1 x_2$$

$$X_{x_4} = e_2 x_0^2 + f_2 x_1^2 + g_2 x_2^2 + h_2 x_0 x_1 + l_2 x_0 x_2 + m_2 x_1 x_2$$

$$X_{x_5} = e_3 x_0^2 + f_3 x_1^2 + g_3 x_2^2 + h_3 x_0 x_1 + l_3 x_0 x_2 + m_3 x_1 x_2 .$$

The tangent hyperplanes are the same if and only if

$$X_{x_3} : Q_{x_3} = X_{x_4} : Q_{x_4} = X_{x_5} : Q_{x_5} \qquad \text{or}$$

(4.3) $$x_2 X_{x_4} + x_1 X_{x_3} = 0 \qquad x_2 X_{x_5} - x_0 X_{x_3} = 0 .$$

The solutions of system (4.3) are the 9 intersection points of two cubic plane curves. To obtain the singular points of V_1 we do not consider the intersection points of the line $x_2 = 0$ and of the conic $X_{x_3} = 0$. So we have only 7 singular points; it is easy to see that, in general, they are ordinary double points.

Now we show the following

Proposition 4.4 (see [4]$_1$). *Let V_1' the blowing up of V_1 along P^1; V_1' is a conic bundle over $\mathbb{P}^2$; its discriminant locus is a degree 7 smooth curve C_1.*

Therefore V_1 is not rational (see $[3]_1$).

Proof. We consider the plane π in $\mathbb{P}^5$ whose equations are: $X_0 = x_1 = x_2 = 0$. π and P^1 are skew. We project V_1 from P^1 on π and we call f such projection. For every point A on π, we indicate by the symbol $\langle A,\ P^1\rangle$ the 3-dimensional linear space generated by A and P^1. The intersection of $\langle A,\ P^1\rangle$ and Q is a quadric surface which splits into P^1 and into another plane $P^{1\prime}$; the intersection of $\langle A,\ P^1\rangle$ and X is a cubic surface which splits into P^1 and into a quadric surface Q'. So that the fibre $f^{-1}(A)$ is given by P^1 and the conic $P^{1\prime} \cap Q'$. If we blow up V_1 along P^1, we obtain a conic bundle V_1'.

Now we calculate the degree of the discriminant curve C_1 on π. The coordinates of A are: $(0:0:0:x_3:x_4:x_5)$. The generic point of $\langle A,\ P^1\rangle$ has coordinates: $(\alpha:\beta:\gamma:\delta x_3:\delta x_4:\delta x_5)$. This point lies on V_1 if and only if

$$(4.5)_a \qquad \alpha\delta x_5 - \beta\delta x_4 + \gamma\delta x_3 = 0$$

$$(4.5)_b \qquad \alpha^2 \delta E_1 + \beta^2 \delta F_1 + \gamma^2 \delta G_1 + \alpha\beta\delta H_1 + \alpha\gamma\delta L_1 + \beta\gamma\delta M_1 + \alpha\delta^2 N_2 + \beta\delta^2 P_2 + \gamma\delta^2 Q_2 + \delta^3 R_3 = 0\,.$$

If $\delta = 0$ we have P^1. If we delete δ, the equations (4.5) are the equations of a conic ψ of $\langle A,\ P^1\rangle$ which is the intersection of the plane $(4.5)_a$ with the quadric $(4.5)_b$. A belongs to C_1 if and only if this conic is degenerated this happens if and only if the projection of ψ over the plane $\alpha = 0$, for example, is degenerated; and this happens if and only if

$$(4.6) \qquad R_3[x_3^2(4E_1F_1 - H_1^2) + x_4^2(4E_1G_1 - L_1^2) + x_5^2(4F_1G_1 - M_1^2) + 2x_3x_4(2E_1M_1 - H_1L_1) - 2x_3x_5(2F_1L_1 - H_1M_1) + 2x_4x_5(2H_1G_1 - L_1M_1)] - E_1(x_3P_2 + x_4Q_2)^2 - F_1(x_3N_2 - x_5Q_2)^2 - G_1(x_4N_2 + x_5P_2)^2 - M_1(x_4N_2 + x_5P_2)(x_3N_2 - x_5Q_2) + H_1(x_3P_2 + x_4Q_2)(x_3N_2 - x_5Q_2) + L_1(x_4N_2 + x_5P_2)(x_3P_2 + x_4Q_2) = 0\,.$$

(4.6) is the equation of C_1 and it has degree 7.

By Proposition 1.2 of $[3]_1$ this curve is smooth because it is easy to see that, for a generic V_1, rank $(\psi) \geqslant 2$.

5 - Conic bundle structure arising from V_n, $n \geqslant 2$

Now we consider V_n, $n \geqslant 2$. By Proposition 4.4 it exists a singular conic bundle V'_n over $\mathbb{P}^2$ with a degree 7 discriminant curve C_n: the blowing up of V_n along P^1. V'_n is singular because the previous calculation shows that there are double ordinary points on every $P^2, \ldots, P^n$. We call f'_n the morphism from V'_n to $\mathbb{P}^2$ whose fibres are conics.

The existence of $P^2, \ldots, P^n$ in V_n implies that C_n splits in the following way $C_n = \Gamma_{8-n} \cup L_1 \cup L_2 \cup \ldots \cup L_{n-1}$, where Γ_{8-n} is a degree $8-n$ smooth curve and $L_1, L_2, \ldots, L_{n-1}$ are $n-1$ lines in generic position.

In fact if in V_1 there is an other plane P^j, it generates a hyperplane of $\mathbb{P}^5$ with P^1; this hyperplane cuts π in a line L_j. L_j is a component of C_n because for every point A of L_j, $\langle A, P^1\rangle$ cuts a line, through the point $P^1 \cap P^j$, on P^j. This line is contained in Q and in X and so it belongs to te fibre $f_n'^{-1}(A)$. Whence $f_n'^{-1}(A)$ splits into this line and into another line meeting the former one.

Every plane in V_1, different from P^1, implies the existence of a line in C_n; so it is clear that an irreducible V_1 can have at most 7 planes other than P^1. In this way we have proved that $n \leqslant 8$.

Now we prove the following

Proposition 5.1. *On every plane P^j in V_n there are 7 ordinary double points; among these points, $n-1$ correspond to the intersections of P^j with the other $n-1$ planes of V_n.*

Blowing up V_n along P^1, on every plane P^j the double point $P^j \cap P^1$ disappear. The remaining 6 points project as follows: the $n-2$ intersections of P^j with the other planes fall in the $n-2$ intersections of L_j with the other lines of C_n; the other $8-n$ fall in the intersections of L_j with Γ_{8-n}.

Proof. By the configuration of the planes P^j, it is enough to verify Proposition 5.1 when in $Q \cap X$ there are only two planes P^1 and P^2. We may suppose that P^2 corresponds to the star of lines centered in $(0:1:0:0)$; by (4.1) its equations are: $x_1 = x_2 = x_5 = 0$.

Then the equation of X is ($e = e_3$, see (4.2))

$$e x_0^2 x_5 + x_1^2 F_1 + x_2^2 G_1 + x_0 x_1 H_1 + x_0 x_2 L_1 + x_1 x_2 M_1$$

$$+ x_0 x_5 N_1 + x_1 P_2 + x_2 Q_2 + x_5 R_2 = 0 .$$

By the same technique used to determine the double points of V_1 on P^1, we have that the double points of V_2 on P^2 $(x_0 : x_3 : x_4)$ are among the 9 intersections of these curves

$$x_3(x_0 H_1 + P_2) + x_4(x_0 L_1 + Q_2) = 0 \tag{5.2$_\mathrm{a}$}$$

$$x_0(x_0 H_1 + P_2) + x_4(e x_0^2 + x_0 N_1 + R_2) = 0 \tag{5.2$_\mathrm{b}$}$$

where $H_1 = H_1(x_3 : x_4 : 0)$ etc.

As usual we have to delete the intersections of the line $x_4 = 0$ and of the conic $x_0 H_1 + P_2 = 0$. One of these points is $P^1 \cap P^2$ and its coordinates on P^2 are $(1:0:0)$. But this point is an ordinary double point for the cubic $(5.2)_\mathrm{a}$; so it is one of the double points of V_2 on P^2.

Now we analyze the double covering of C_n. The hyperplane generated by P^1 and P^2 cuts the line L_1 on $\pi(x_3 : x_4 : x_5)$; the equation of L_1 is $x_5 = 0$; $C_2 = \Gamma_6 \cup L_1$ and the equation of Γ_6 on π is

$$\begin{aligned}
R_2[x_3^2(4ex_5 F_1 - H_1^2) + x_4^2(4ex_5 G_1 - L_1^2) + x_5^2(4F_1 G_1 - M_1^2) + 2x_3 x_4(2ex_5 M_1 - H_1 L_1) \\
- 2x_3 x_5(2F_1 L_1 - H_1 M_1) + 2x_4 x_5(2H_1 G_1 - L_1 M_1)] - e(x_3 P_2 + x_4 Q_2)^2 \\
- x_5 F_1(x_3 N_1 - Q_2)^2 - x_5 G_1(x_4 N_1 + P_2)^2 - x_5 M_1(x_4 N_1 + P_2)(x_3 N_1 - Q_2) \\
+ H_1(x_3 P_2 + x_4 Q_2)(x_3 N_1 - Q_2) + L_1(x_4 N_1 + P_2)(x_3 P_2 + x_4 Q_2) = 0\,.
\end{aligned}$$

The generic point $A(s)$ of L_1 has non homogeneous coordinates $(0:0:0:s:1:0)$ in $\mathbb{P}^5$. $\langle A,\ P^1\rangle$ has equations: $x_5 = x_3 - s x_4 = 0$.

Let us intersect $\langle A,\ P^1\rangle$ with V_2 and obtain

$$x_5 = 0 \qquad x_3 = s x_4 \qquad x_1 x_4 - s x_2 x_4 = 0$$

$$x_1^2 F_1 + x_2^2 G_1 + x_0 x_1 H_1 + x_0 x_2 L_1 + x_1 x_2 M_1 + x_1 P_2 + x_2 Q_2 = 0$$

where $F_1 = F_1(s x_4 : x_4 : 0)$ etc.

If $x_4 = 0$ we have the equations of P^1. Otherwise we have the conic

$$x_5 = 0 \qquad x_3 = s x_4 \qquad x_1 = s x_2$$

$$x_2(s^2 x_2 F_1 + x_2 G_1 + s x_0 H_1 + x_0 L_1 + s x_2 M_1 + s x_4 P_2 + x_4 Q_2) = 0$$

where $F_1 = F_1(s:1:0)$ etc. This conic splits into two lines: $l(s)$, joining $A(s)$ with $P^1 \cap P^2$ and $r(s)$, whose equations are respectively

$$x_1 = x_2 = x_5 = 0 \qquad x_3 = sx_4$$

$$x_5 = 0 \qquad x_3 = sx_4 \qquad x_1 = sx_2$$

$$x_0(sH_1 + L_1) + x_2(s^2F_1 + G_1 + sM_1) + x_4(sP_2 + Q_2) = 0\,.$$

Whence $f^{-1}(L_2) = P^2 \cup \bar{P}^2$, where $\bar{P}^2$ is the ruled surface generated by $r(s)$. This surface intersects P^2 along a curve τ, whose equation on P^2 is: $x_0 x_3 H_1 + x_0 x_4 L_1 + x_3 P_2 + x_4 Q_2 = 0$, where $H_1 = H_1(x_3 : x_4 : 0)$ etc.
τ is the same curve $(5.2)_a$ on which there are the double points of V_2 on P^2. The line $r(s)$ intersects τ in the same point at which $l(s)$ meets τ other than $P^1 \cap P^2$. It is easy to verify that the conditions under which $A(s)$ is a double point of C_2 (i.e. it is the intersection of L_2 with Γ_6) are the same conditions under which $l(s)$ passes through a double point of V_2 on P^2 different from $P^1 \cap P^2$. This fact proves that the double points of V_2' project by f_2' into the double points of C_2.

Remark 5.3. The family of lines of the singular fibres of f_2' becomes a curve $\tilde{C}_2$. The previous proof also shows that $\tilde{C}_2$ splits into a smooth curve $\tilde{\Gamma}_6$ and into a couple of rational curves $\tilde{L}_{1,1}$ and $\tilde{L}_{1,2}$ one of which is a line. $\tilde{\Gamma}_6$ is an unramified double covering of Γ_6 and $\tilde{L}_{1,1} \cup \tilde{L}_{1,2}$ is an unramified double covering of L_1: $\tilde{L}_{1,1}$ does not intersects $\tilde{L}_{1,2}$ while $\tilde{\Gamma}_6$ intersects $\tilde{L}_{1,i}$ $(i = 1, 2)$ transversally in 6 points; these points project by f_2' into the intersection points of Γ_6 with L_1 so that the double covering $\tilde{f}_2' : \tilde{C}_2 \to C_2$, induced by f_2', is unramified (it is not a «pseudorevêtement» according to Beauville (see $[3]_1$), in spite of the singular C_2). If $n \geq 3$ we have $\tilde{C}_n = \tilde{\Gamma}_{8-n} \cup \tilde{L}_{1,1} \cup \tilde{L}_{1,2} \cup \ldots \cup \tilde{L}_{n-1,1} \cup \tilde{L}_{n-1,2}$ and $\tilde{C}_n$ is always an unramified double covering of C_n with the characteristics above explained.

Remark 5.4. The conic bundle structures we have studied for $n \geq 3$ are nothing else than conic bundle structures arising from cubic threefolds in $\mathbb{P}^4$: it is easy to see this by using the birational map Φ described in 3.

We may always suppose that the third plane in V_n has equations: $x_0 = x_2 = x_4 = 0$, it corresponds to the star of lines centered in $(0:0:1:0)$. In this case the line L_2 on π has equation: $x_4 = 0$. Now we look at Φ and we put $W_n = \Phi^{-1}(V_n)$, $n \geq 3$. W_n is a cubic hypersurface of $\mathbb{P}^4$, it contains l_1, l_2 and l_3 (because V_n contains π_{12}, π_{13}, π_{23}) and $n-3$ planes. If we project W_n from l_1 to the skew plane π', whose equations are: $z_1 = z_2 = 0$, and we blow up W_n along l_1, we obtain a conic bundle structure which is well-known when $n = 3$ (see [4]$_1$). By our suitable choice of coordinate system it is easy to see, by direct calculation, that the discriminant locus in π' is a curve D_n which is exactly $C_n \setminus (L_1 \cup L_2)$ if we set $z_i = x_i$, $i = 3, 4, 5$. What we have proved for V_n and C_n is also true for W_n and D_n: in fact these conic fibrations are birationally equivalent (see [**10**]).

6 - The cases $n = 7$ and $n = 8$

We recall that V_8 is the rational cubic complex of lines lying on the quadrics of a net in $\mathbb{P}^3$. Now we prove the following

Proposition 6.1. *Let us choose 7 generic points $B_1, B_2, \ldots, B_7$ in $\mathbb{P}^3$, let us consider V_7 containing the 7 planes corresponding to the 7 stars of lines centered in $B_1, B_2, \ldots, B_7$. V_7 contains another plane which corresponds to the stars of lines centered in the univocally determined point B_8 such that $B_1, B_2, \ldots, B_8$ are the base locus of a net of quadrics in $\mathbb{P}^3$.*

Corollary 6.2. *If we choose 8 generic points in $\mathbb{P}^3$ and we look for a cubic X containing the 8 planes corresponding to the 8 stars of lines centered in these 8 points, we have that X splits into Q and a hyperplane.*

Proof. We can prove our thesis directly by calculation, but we prefer to use a synthetic reasoning which is substantially contained in [4]$_2$.

Let us suppose $B_1, B_2, \ldots, B_7$ and V_7 fixed. We consider the net of quadrics determined by $B_1, B_2, \ldots, B_7$; this net has another base point B_8. We call M the cubic complex of lines lying on the quadrics of this net. It is sufficient to show that $V_7 = M$. On the contrary suppose that $V_7 \neq M$. We fix a quadric R of the net such that R does not contain the lines joining B_i with B_j $(i, j = 1, \ldots, 8)$. The two rulings of R determine two conics c_1 and c_2 in $G(1, 3) = Q$. Since $V_7 \neq M$ and since R is generic in the net, c_1 is not contained in V_7. Let $\Omega(1, 3)$ the generator of the equivalence class of 3-dimensional cycles of Q in $H^*(Q, Z)$ (see [**6**]). Let $\Omega(0, 2)$ the analogous generator for 1-dimensional cycles, and $\Omega(0, 1)$ for 0-dimensional ones. Then $V_7 = 3\Omega(1, 3)$ and $c_1 = 2\Omega(0, 2)$ in $H^*(Q, Z)$; so that $V_7 \cdot c_1 = 6\Omega(0, 1)$. It means that there are 6 lines common to V_7 and to the ruling c_1. But there are at least 7 lines common to V_7 and the ruling c_1: the 7 lines of c_1 going through the 7 points $B_1, B_2, \ldots, B_7$ of R. This is a contradiction.

7 - The cases $4 \leqslant n \leqslant 6$

By previous sections we see that we can apply Beauville's theory to V_n, $n \geqslant 2$ (see $[3]_1$, § 2.9, p. 335). In our case the double covering $\bar{f}'_n : \tilde{C}_n \to C_n$ is unramified and C_n is smooth except for the intersections of its components; therefore $\bar{\bar{f}}'_n$ is unramified and N_n is isomorphic to the disjoint union of Γ_{8-n} and $n-1$ copies of $\mathbb{P}^1$, $\tilde{N}_n$ is isomorphic to the disjoint union of $\tilde{\Gamma}_{8-n}$ and $2(n-1)$ copies of $\mathbb{P}^1$.

If we call V''_n the blowing up of V'_n at its ordinary double points, we have $\mathrm{Prym}(\tilde{N}_n, N_n) \simeq \mathrm{Prym}(\tilde{\Gamma}_{8-n}, \Gamma_{8-n}) \simeq J(V''_n)$.

Remark 7.2. If $n=2$, the well known Mumford theorem (see [8] and $[3]_2$) says that $J(V''_2)$ is not the product of Jacobians, so that V''_n and V_2 are not rational. If $n \geqslant 3$, $J(V''_n)$ may be the Jacobian of a curve (and if $n \geqslant 4$ that is the case), therefore the theory of Prym varieties does not prove the rationality or irrationality of V_n.

We can apply Theorem 1.13 of [10] (Prop. 1.16 and 1.17) to $(V'_n, \mathbb{P}^2, f'_n)$, $n \geqslant 2$.

If we blow up $\mathbb{P}^2$ at the singular points of C_n, we have a smooth rational surface H and a birational map $\sigma : H \to \mathbb{P}^2$. By Sarkisov's theorem there exists the following commutative diagram

$$\begin{array}{ccc} V'_n & \xleftarrow{\ \lambda\ } & V'''_n \\ {\scriptstyle f'_n}\downarrow & & \downarrow{\scriptstyle f'''_n} \\ \mathbb{P}^2 & \xleftarrow{\ \sigma\ } & H \end{array}$$

where λ is a birational map and (V'''_n, H, f'''_n) is a regular c.f. whose discriminant locus is the proper transformed of C_n by σ, i.e. the disjoint union of $\sigma^*(\Gamma_{8-n})$ and of $\sigma^*(L_i)$, $i = 1, \ldots, n-1$.

By construction (V'''_n, H, f'''_n) is a regular c.f.; but it is not standard because $f'''^{-1}_n(\sigma^*(L_i))$, $i = 1, \ldots, n-1$, splits into a couple of irreducible surfaces, each of them is a $\mathbb{P}^1$-bundle.

Sarkisov's Theorem assures that blowing down one of these $\mathbb{P}^1$-bundle, we can obtain a standard c.f. (V''''_n, H, f''''_n), birationally equivalent to the previous one; now the discriminant locus is only $\sigma^*(\Gamma_{8-n})$ (see also [9], Prop. 6.3). Finally we can embed (V_n''''', H, f_n''''') in a projective space to obtain $(V_n\#, H, f_n\#)$ which is a conic bundle according to Def. 2.1 (see [10], 1.5).

V_n is rational if and only if $V_n\#$ is rational. $(V_n\#, H, f_n\#)$ is a standard conic bundle, so we can apply Theorem 1 of $[7]_2$.

Let $n=4$. Γ_4 is a smooth plane quartic; let us fix a point of Γ_4 and consider the pencil of lines $\{L_\nu\}$ going through it. Then the pencil of curves $\{C_\nu\}=\{\sigma^*(L_\nu)\}$ satisfies the hypothesis of Theorem 1 of $[7]_2$ for the standard conic bundle $(V_4\#, H, f_4\#)$. So $V_4\#$ and V_4 are rational.

Let $n=5$. We obtain the rationality of V_5 as in the case $n=4$.

Let $n=6$. $C_6=\Gamma_2\cup L_1\cup\ldots\cup L_5$. Since the double covering of C_6 is always an unramified covering, $\tilde{\Gamma}_2$ must split into a couple of rational curves.

Then $f_6'''^{-1}(\sigma^*(\Gamma_2))$ splits too and when we consider $(V_6\#, H, f_6\#)$ the component $\sigma^*(\Gamma_2)$ disappears from the discriminant locus: it reduces to $\emptyset$. The rationality of V_6 follows from Iskovskih's theorem.

Finally we remark that it is possible to prove the rationality of V_7 and V_8 as in the case $n=6$.

8 - Case $n=3$ and Iskovskih's conjectures

Recently Iskovskih has made the following two Conjectures 8.1 and 8.3 about the rationality of conic bundles.

Conjecture 8.1. (See $[7]_{3,4}$). Let $h: V\to S$ be a standard conic bundle over the rational surface S, with a connected discriminant curve C ($C\neq\emptyset$). Then V is rational if and only if one of the following assertions holds:

(i) There exists a pencil of rational curves $\{C_\nu, \nu\in \mathbf{P}^1\}$, having no fixed components on S such that $C_\nu\cdot C\leqslant 3$ for every ν (i.e. $\{C_\nu\}$ defines a rational map $\eta: C\to\mathbf{P}^1$ whose degree does not exceed 3).

(ii) There exists a birational map $\rho: S\to\mathbf{P}^2$ such that $\rho(C)$ is a curve of degree 5, which has at most ordinary double points, and such that for the double covering $\bar{h}:\overline{\rho(C)}\to\rho(C)$, induced by h, the condition $h^0(\overline{\rho(C)}, \bar{h}^*(L))=3$ is fulfilled, where L is a hyperplane divisor of $\mathbf{P}^2$.

This conjecture has been proved when $S=\mathbf{P}^2$ or S is a $\mathbf{P}^1$-bundle over a rational curve (see [11]); the *if* part of the conjecture is always true.

Remark 8.2. If $4\leqslant n\leqslant 7$ $(V_n\#, H, f_n\#)$ satisfies condition (i) of 8.1. If

$n = 3$ ($V_3\#$, H, $f_3\#$) satisfies condition (ii) of 8.1 with $\rho = \sigma$ and $\rho(C) = \Gamma_5$ and we have $h^0(\tilde{\Gamma}_5, \overline{f_3\#}^*(L)) = h^0(\tilde{\Gamma}_5, \overline{f_3''}^*(L))$. Moreover $h^0(\tilde{\Gamma}_5, \overline{f_3''}^*(L)) = 3$ if and only if the θ-characteristic $L + N$ is even, where N is the point of order two, in the Jacobian of Γ_5, corresponding to the unramified covering considered before (see [11]). This condition is also necessary and sufficient to guarantee that $J(V_3')$ is isomorphic to the Jacobian of a curve, according to Beauville's theory (see $[3]_2$); actually the θ-characteristic $L + N$ is odd as V_3 is not rational.

Conjecture 8.3. (See $[7]_3$). Under the hypothesis of Conjecture 8.1, V is rational if and only if one of the two following conditions holds

(1) $|2K_S + C| = \emptyset$ (2) condition (ii) of Conjecture 8.1

(when it exists a birational map $\rho: S \to \mathbb{P}^2$ such that $\rho(C)$ is a curve of degree 5, which has at most ordinary double points, you have to check condition (2)).

This conjecture has been proved for $S = \mathbb{P}^2$ or S a $\mathbb{P}^1$-bundle over a rational curve (see [11]).

Remark 8.4. What we have proved in **7** agrees with 8.3: in fact we can apply Conjecture 8.3 to ($V_n\#$, H, $f_n\#$), with $n \geq 1$, $n \neq 3$, and we obtain: $K_{\mathbb{P}^2} \equiv -3L$, $K_H \equiv -3\sigma^*(L) + E$, where «$\equiv$» means linear equivalence and E is the sum of all the exceptional divisors introduced by σ. Then $C \equiv (8-n)\sigma^*(L) - 2E$ and so $|2K_H + C| = |(2-n)\sigma^*(L)| = \emptyset$ if and only if $n \geq 4$. If $n = 3$ we have to check condition (2), which is not fulfilled as we said before.

Remark 8.5. The quoted results of Iskovskih and Beauville allow us to prove the following theorem (see [11]).

Theorem 8.6. *Let $f: V \to \mathbb{P}^2$ be an ordinary (hence standard) conic bundle. Then V is rational if and only if $J(V)$ is isomorphic to the Jacobian of a curve.*

References

[1] A. Alzati and M. Bertolini, *On the problem of rationality for some cubic complexes*, Indag. Math. **91** (1988), 349-364.

[2] E. Ambrogio e D. Romagnoli, *Sulla non razionalità della varietà di Fano di IP^5 contenente 2 o 3 piani*, Quaderno del Dipartimento di Matematica dell'Università di Torino (1987).

[3] A. BEAUVILLE: [•]$_1$ *Varietés de Prym et Jacobiennes intermédiaires*, Ann. Sci. École Norm. Sup. (4) 10 (1977), 309-391; [•]$_2$ *Prym varieties and the Schottky Problem*, Invent. Math. 41 (1977), 149-196.

[4] A. CONTE: [•]$_1$ *Introduzione alle varietà algebriche a tre dimensioni*, Quaderno U.M.I. 22, Ed. Pitagora Bologna (1982); [•]$_2$ *Two examples of algebraic threefolds whose hyperplane sections are Enriques surfaces*, «Algebraic threefolds», Varenna (1981), ed. A. Conte S.L.N. n. 947.

[5] G. FANO, *Sulle varietà algebiche a tre dimensioni aventi tutti i generi nulli*, Atti del Congresso di Bologna, ed. Zanichelli, 1928.

[6] S. KLEIMAN and D. LAKSOV, *Schubert Calculus*, Amer. Math. Monthly 79 (1972), 1061-1082.

[7] V. A. ISKOVSKIH: [•]$_1$ *Fano threefolds* (II), Izv. Akad. Nauk. S.S.S.R. 41 (1977), 516-562; [•]$_2$ *Congruences of conics in* IP^3, Vestnik Mosk. Univ. Mat. (37) 6 (1982), 57-62; [•]$_3$ *Algebraic Threefolds with special regard to the Problem of Rationality*, Proc. of the Internat. Congr. of Math., Warszava, 1983; [•]$_4$ *On the rationality problem for conic bundles*, Duke Math. J. 54 (1987), 271-294.

[8] D. MUMFORD, *Prym varieties* (I), «Contribution to Analysis: a collection of papers dedicated to Lipman Bers», New York (1974), 325-350.

[9] S. MORI and S. MUKAI, *On Fano 3-folds with* $B_2 \geqslant 2$, Advanced Studies in Pure Mat. 1 (1983) Alg. Var. and Analytic Var., 101-129.

[10] V. G. SARKISOV, *On conic bundle structures*, Math. U.S.S.R. Izvestiya (20) 2 (1982).

[11] V. V. SHOKUROV, *Prym varieties: theory and applications*, Math. U.S.S.R. Izvestiya (23) 1 (1984).

Sommario

Si prova la razionalità di una classe di fibrati in coniche su superfici razionali utilizzando alcuni recenti risultati di Sarkisov ed Iskovskih. Tali fibrati provengono dall'intersezione di una iersuperficie quadrica liscia e di una ipersuperficie cubica liscia in IP^5, *contenente piani che si incidono due a due in un solo punto; essi sono birazionalmente equivalenti a fibrati in coniche provenienti da ipersuperfici cubiche di* IP^4. *Si mostra inoltre che* 8 *è il massimo numero di tali piani che le due ipersuperfici possono contenere senza spezzarsi.*

Le nostre conclusioni costituiscono una conferma di una celebre congettura di Iskovskih sulla razionalità dei fibrati in coniche.

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