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On a generalization of the Bessel-Clifford equation and an application in quantum mechanics (**)

1 - Introduction

The differential equation

$$xy'' + cy' - y = 0 \tag{1.1}$$

known as the Bessel-Clifford equation, and which is associated with the hypergeometric function ${}_0F_1(-;\ c;\ x)$, is closely related to Bessel's equation and has attracted some attention in its own right. See, for example [3]. In this study, a generalised form of (1.1) is considered, namely,

$$xy'' + cy' - y = k^{-h} x^{h-1} y \tag{1.2}$$

which includes the additional term on the right. It is taken that h and c are real, and for the moment, no further restrictions are imposed.

If h is a positive integer greater than unity, (1.2) has an irregular singularity of the h^{th} species at infinity, while the other singularity at the origin remains regular.

The solution of (1.2) is attempted in the form of a series involving the parameter k, and in order that this series may be convergent, we put

$$x = kh^{2/h} z^{1/h} \tag{1.3}$$

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and obtain

$$zy'' + (1 + \frac{c-1}{h})y' - y = pz^{-1+1/h}y \tag{1.4}$$

where $p = kh^{-2+2/h}$.

The classical means of solution of a linear differential equation is the Frobenius method ([4], p. 396), but in the case under consideration, difficulties arise in constructing the general solution of the associated recurrence relation. We thus attempt a tentative solution of (1.4) as a power series in the parameter p in the form

$$y = \sum_{r=0}^{\infty} p^r y_r(z). \tag{1.5}$$

The question of the convergence of such a series must be investigated in each particular case, since no general theory has so far been developed.

2 - The formal development of the series (1.5)

If the series (1.5) is substituted into (1.4), it is clear that

$$zy_0'' + (1 + \frac{c-1}{h})y_0' - y_0 = 0 \tag{2.1}$$

$$zy_r'' + (1 + \frac{c-1}{h})y_r' - y_r = z^{-1+1/h}y_{r-1} \qquad r = 1, 2, 3, \ldots. \tag{2.2}$$

One suitable form of y_0 is thus

$$y_0 = {}_0F_1(-;\ 1 + \frac{c-1}{h};\ z) \tag{2.3}$$

and so y_1 is determined by

$$zy_1'' + (1 + \frac{c-1}{h})y_1' - y_1 \tag{2.4}$$

$$= z^{-1+1/h}y_0 = \sum_{m=0}^{\infty} \frac{z^{m-1+1/h}}{(1 + \frac{c-1}{h},\ m)m!}.$$

As usual, the Pochhammer symbol $(a,\ m)$ is given by

$$(a,\ m)=a(a+1)(a+2)\ \dots\ (a+m+1)=\Gamma(a+m)/\Gamma(a) \qquad (a,\ 0)=1. \tag{2.5}$$

(See, for example [2], p. 14.) Hence,

$$y_1=\sum_{m=0}^{\infty}\frac{{}_0f_{1,m+1/h}(-;\ 1+\frac{c-1}{h};\ z)}{(1+\frac{c-1}{h},\ m)\,m!} \tag{2.6}$$

and the inhomogeneous hypergeometric function ${}_0f_{1,m+1/h}(-;\ 1+\frac{c-1}{h};\ z)$ may be written

$$\frac{z^{m+1/h}}{(m+1/h)(m+c/h)}\sum_{n=0}^{\infty}\frac{z^n}{(m+1+1/h,\ n)(m+1+c/h,\ n)}. \tag{2.7}$$

(See [1], p. 277.) After a little reduction, it is found that

$$y_1=\frac{h^2 z^{1/h}}{c}\,{}_2F_1(1/h,\ c/h;\ 1+c/h-1/h;\ 1)\,{}_1F_2(1;\ 1+1/h,\ 1+c/h;\ z). \tag{2.8}$$

In order that this series should converge, it is necessary that $h>2$. Similarly,

$$y_2=\frac{h^4 z^{2/h}}{2!\ c(c+1)}\,{}_2F_1(1/h,\ c/h;\ 1+c/h-1/h;\ 1) \tag{2.9}$$

$$\times\,{}_3F_2(2/h,\ c/h+1/h,\ 1;\ 1+c/h,\ 1+1/h;\ 1)\,{}_1F_2(1;\ 1+2/h,\ 1+c/h+1/h;\ z).$$

If we write

$$F_n={}_3F_2(n/h+1/h,\ c/h+n/h,\ 1;\ 1+c/h+n/h-1/h,\ 1+n/h;\ 1) \tag{2.10}$$

$$G_n={}_1F_2(1;\ 1+n/h,\ 1+c/h+n/h-1/h;\ z) \tag{2.11}$$

then

$$y_0=G_0 \tag{2.12}$$

$$y_r=\frac{h^{2r}z^{r/h}}{r!\ (c,\ r)}\Big[\prod_{n=0}^{r-1}F_n\Big]G_r \qquad r=1,\ 2,\ 3,\ \dots. \tag{2.13}$$

Returning to the original notation, we see that

(2.14) $$G_n = {}_1F_2(1;\ 1+n/h,\ 1+c/h+n/h-1/h;\ \frac{x^h}{k^h h^2})$$

(2.15) $$p^r y_r = \frac{x^r}{r!\ (c,\ r)}[\prod_{n=0}^{r-1} F_n]\, G_r .$$

With the above form of y_r, we denote the series

(2.16) $$\sum_{r=0}^{\infty} k^r y_r \quad \text{by} \quad y(k,\ c,\ h;\ x).$$

If y is replaced by $x^{1-c}y$ in (1.2), we obtain

(2.17) $$xy'' + (2-c)\,y' - y = k^{-h} x^{h-1} y .$$

Hence, a second independent solution of (1.2) is

(2.18) $$x^{1-c} y(k,\ 2-c,\ h;\ x).$$

3 - The convergence of the series (1.5)

The ratio of two successive terms of (1.5) may be written

(3.1) $$p y_{r+1}/y_r = \frac{ph^2 z^{1/h}}{(1+r)(c+r)} F_r G_{r+1}/G_r \qquad r+1,\ 2,\ 3,\ \dots .$$

For sufficiently large values of r, it is clear from (2.11) that $G_{r+1} < G_r$ because $h > 2$. Furthermore, since F_r as given by (2.10) is a convergent series, it is finite, so that

(3.2) $$\lim_{r\to\infty} \frac{py_{r+1}}{y_r} = 0$$

and (1.5) is convergent.

The series (1.5) may similarly be shown to be absolutely convergent for all finite values of its parameters and variable, with the sole restriction that $h > 2$.

Since (1.5) consists of a convergent series of power series, it is also uniformly convergent under the same conditions.

4 - The asymptotic behaviour of (1.2)

If the determining factor of $\exp(\pm 2z^{1/2})$ is removed from (1.4), then we have the subnormal solutions

$$\exp(\pm 2z^{1/2})\, z^{(1/h-c/h-1/2)/2} \tag{4.1}$$

or, for (1.2),

$$\exp(\pm 2x^{1/2} k^{-h/2} h^{-1})\, x^{(1-c-h/2)/2}. \tag{4.2}$$

In connection with the application which is discussed in **5** below, the asymptotic behaviour of the principal solution of (1.2) for large positive real values of x is required. From (2.11) it follows that

$$\begin{aligned} G_r &= {}_1F_2(1;\ 1+r/h,\ 1+c/h+r/h-1/h;\ z) \\ &\sim \Gamma(1+r/h)\Gamma(1+c/h+2r/h-1/h)\,\pi^{1/2}\exp(2z^{1/2})\, z^{(1/h-c/h-1/2-2r/h)/2} \end{aligned} \tag{4.3}$$

for $z \to +\infty$ (see [5], p. 1172). Hence

$$\begin{aligned} \sum_{r=0}^{\infty} p^r y_r &\sim \pi^{1/2}\Gamma(1+c/h-1/h+2r/h)\Gamma(1+r/h) \\ &\times \Big[\prod_{n=0}^{r-1} F_n\Big] \frac{h^{c/h+2r/h-1/h+1/2}\, k^{r+1/2-c/2-h/4}}{r!\ (c,\ r)}\, x^{(1-c-h/2)/2} \exp(2x^{h/2} k^{-h/2} h^{-1}) \end{aligned} \tag{4.4}$$

and the required asymptotic form of $y(k,\ c,\ h;\ x)$ then follows.

5 - An application in the theory of quantum mechanics

The quantum motion of an anharmonic oscillator in a centrally symmetric field with potential αr^{λ}, where r is the radial coordinate, is governed by the differential equation

$$R'' + [\frac{2\mu}{h^2}(E - \alpha r^{\lambda}) - \frac{l(l+1)}{r^2}]R = 0 \tag{5.1}$$

where l is the azimuthal quantum number. The differential equation (1.2) may be put into this form if x is replaced by $-qr^2$ and we have

$$R'' - [\frac{(1-2c)(3-2c)}{4r^2} - 4q + 4(-q/k)^h r^{2h-2}]R = 0 \tag{5.2}$$

so that explicit forms of the eigenfuctions can be obtained. The eigenvalues are the values of k for which (4.4) is zero, $c = 1/2 - 2l$ and $4(-q/k)^h = -\alpha$, recalling that $h > 2$, or $\lambda > 2$. In general, a problem of this type for general values of λ is tackled by approximate methods.

References

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Summary

A term involving a simple power of the independent variable is added to the Bessel-Clifford equation and a closed-form convergent series solution in terms of a parameter is obtained by the application of inhomogeneous hypergeometric functions. An asymptotic form of this solution valid for large positive values of the independent variable is deduced and the result applied to a type of anharmonic quantum oscillator.

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