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The chromatic number of some graphs in the euclidean plane (**)

Let $M(x_1, y_1)$ and $N(x_2, y_2)$ be two points in the Euclidean plane $\mathbb{E}^2$. It is well-known that the following definitions yield metrics for the Euclidean plane:

(a) $d(\mathrm{M}, \mathrm{N}) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ euclidean distance

(b) $d_4(\mathrm{M}, \mathrm{N}) = |x_1 - x_2| + |y_1 - y_2|$ city block distance

(c) $d_8(\mathrm{M}, \mathrm{N}) = \max(|x_1 - x_2|, |y_1 - y_2|)$ chessboard distance.

The numbers 4 and 8 are appropriate, because if we restrict ourselves to the points of $\mathbb{E}^2$, which have integral coordinates (*digital points*), these numbers represent the number of points at distance one (the neighbours) from a given point, with respect to these two metrics [3].

We shall define the infinite graphs G, G_4 and G_8, in the following way: the vertex set of these graphs is the set of points of $\mathbb{E}^2$, two vertices being adjacent if and only if their Euclidean, city block, respectively chessboard distance is equal to one.

Theorem. *The chromatic number of the graphs G_4 and G_8 is equal to four, or $\chi(G_4) = \chi(G_8) = 4$.*

Proof. The set of points having coordinates (0, 0), (1/2, 1/2), (1/2, − 1/2) and (1, 0) is a 4-clique for G_4, and the points (0, 0), (0, 1), (1, 0) and (1, 1)

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induce a 4-clique for G_8, which implies that $\chi(G_4) \geq 4$ and $\chi(G_8) \geq 4$. It remains to define a 4-coloring for G_4 and for G_8. In the case of G_4, consider all lines with slope ± 1 passing through the digital points of $\mathbb{E}^2$. The intersection points of these lines are points $M(p, q)$, where $p, q \in \mathbb{Z}$ and points $N(r/2, s/2)$, with $r, s \in \mathbb{Z}$, $r, s \equiv 1 \pmod 2$.

Denote by S the set of these points. Color $M(p, q)$ with the color α, if $p \equiv q \pmod 2$, and with the color β, if $p \equiv q + 1 \pmod 2$.

A point $N(r/2, s/2)$ will be colored with the color γ, if $r \equiv s \pmod 4$, and with the color δ, if $r \equiv s + 2 \pmod 4$. For $P(u, v) \in S$, denote $Q(u - 1/2, v + 1/2)$ and $R(u - 1/2, v - 1/2)$. If P is colored with the color $a \in \{\alpha, \beta, \gamma, \delta\}$, then all interior points of the segments PQ and PR will be also colored with the color a. In this way, any square $ABCD$, having vertices in S and the length of the side equal to $\sqrt{2}/2$, will have the four vertices colored with $\alpha, \beta, \gamma, \delta$ and the colors of the sides will be a, a, b, c, where $a, b, c \in \{\alpha, \beta, \gamma, \delta\}$. In this case, color all interior points of $ABCD$ with the color a. In this way, all points of $\mathbb{E}^2$ are colored with four colors. It is easy to see that if $d_4(E, F) = 1$, then E and F have different colors.

A 4-coloring of G_8 may be defined in a similar manner. Denote by S the set of digital points of $\mathbb{E}^2$. Color the points of S in the following way: $M(p, q)$, with $p, q \in \mathbb{Z}$, will be colored with the color: α if $p \equiv 1 \pmod 2$ and $q \equiv 1 \pmod 2$; β if $p \equiv 0 \pmod 2$ and $q \equiv 0 \pmod 2$; γ if $p \equiv 1 \pmod 2$ and $q \equiv 0 \pmod 2$; δ if $p \equiv 0 \pmod 2$ and $q \equiv 1 \pmod 2$.

If $M(p, q)$ is colored with the color a, then all interior points of the segment MQ and MR, with $Q(p - 1, q)$ and $R(p, q - 1)$, will be also colored with the color a. Any unit square $ABCD$, with its vertices in S, has the four vertices colored with $\alpha, \beta, \gamma, \delta$, and the colors of the sides are $a, a, b, c \in \{\alpha, \beta, \gamma, \delta\}$.

Color all interior points of $ABCD$ with the color a. Now, if $d_8(E, F) = 1$ then the points E and F will have different colors. The proof is complete.

Note that the determination of $\chi(G)$ is an open problem, in the Euclidean Ramsey theory [1] [2]. It is only known that $4 \leq \chi(G) \leq 7$ and it has been conjectured that, also, $\chi(G) = 4$.

References

[1] B. Bollobás, *Graph theory. An introductory course*, Springer-Verlag, New York, 1979.

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Résumé

On donne un théorème de 4-coloration pour certain graphes définis sur le plan euclidien.

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