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ALESSANDRO FEDELI (*)

Two properties related to $\mathcal{A}$-compactness (**)

0. – Let $\mathcal{A}$ be a (non empty) class of topological spaces, let X be a topological space and F a subset of X. A point x of X is said to be a *point* of *$\mathcal{A}$-closure* of F in X if for each $f, g: X \to A$, $A \in \mathcal{A}$, such that $f|F = g|F$ (where $f|F$ denotes the restriction of f to F), $f(x) = g(x)$.

The set of all points of $\mathcal{A}$-closure of F in X is said to be the *$\mathcal{A}$-closure of F in* X and it is denoted by $[F]^X_{\mathcal{A}}$.

This closure operator was introduced by Salbany [13], and studied by Dikranjan and Giuli in $[4]_{1,2}$.

A class of topological spaces is said *epireflective* iff it is closed under the formation of products and subspaces [8]. Each class $\mathcal{B}$ of topological spaces has an epireflective hull $\mathcal{E}(\mathcal{B})$ (i.e. there exists a smallest epireflective subcategory containing $\mathcal{B}$). For every $X \in$ TOP and $M \subset X$ and every $\mathcal{A} \subset$ TOP$[M]^X_{\mathcal{A}} = [M]^X_{E(\mathcal{A})}$ holds (Prop. 1.4, $[4]_1$), hence in the sequel we consider exclusively epireflective subcategories of TOP.

Def. 0.1. Let $\mathcal{A} \subset$ TOP, $X \in$ TOP and $F \subset X$:

(a) F is said to be *$\mathcal{A}$-closed* in X if $[F]^X_{\mathcal{A}} = F$.

(b) A function $f: X \to Y$, $X, Y \in \mathcal{A}$, is *$\mathcal{A}$-continuous* if $f([F]^X_{\mathcal{A}}) \subset [f(F)]^Y_{\mathcal{A}}$, $F \subset X$.

Every continuous function $f: X \to Y$, $X, Y \in \mathcal{A}$, is $\mathcal{A}$-continuous (Prop. 1.2(x), $[4]_1$).

(c) A function $f: X \to Y$, $X, Y \in \mathcal{A}$, is said to be *$\mathcal{A}$-closed* if for every $\mathcal{A}$-closed set $F \subset X$ the image $f(F)$ is $\mathcal{A}$-closed in Y.

(*) Indirizzo: Via Assergi 4, I-67100 L'Aquila.
(**) Ricevuto: 24-I-1989.

(d) The coarsest topology in X which contains all *$\mathcal{A}$-closed* subsets as closed sets is said to be the *$\mathcal{A}$-closure topology of* X and, if τ is the topology of X, it is denoted by $\tau_{\mathcal{A}}$.

$F_{\mathcal{A}}$: TOP $\to$ TOP will denote the functor which assigns to $(X, \tau) \in$ TOP the space $(X, \tau_{\mathcal{A}})$.

For each continuous map $f:(X, \tau) \to (Y, \sigma)$ in TOP the continuity of $f = F_{\mathcal{A}}(f):(X, \tau_{\mathcal{A}}) \to (Y, \sigma_{\mathcal{A}})$ follows from 1.2(x) of $[4]_1$.

$F_{\mathcal{A}}$ is said *finitely multiplicative* if it preserves finite products, i.e. $(\pi_I, \tau_i)_{\mathcal{A}} = \pi_I(\tau_i)_{\mathcal{A}}$, $I = 1, 2, \dots, n$ $[4]_2$.

The $\mathcal{A}$-closure is not in general a Kuratowski operator (cf. [3], $[4]_1$).

If the $\mathcal{A}$-closure is a Kuratowski operator then is easy to see that a function $f:(X, \tau) \to (Y, \sigma)$ is $\mathcal{A}$-continuous ($\mathcal{A}$-closed) iff $f = F_{\mathcal{A}}(f):(X, \tau_{\mathcal{A}}) \to (Y, \sigma_{\mathcal{A}})$ is continuous (closed). In this paper we consider only the $\mathcal{A}$-closures that are Kuratowski operators.

Notation 0.2. The following categories are denoted as follows:

TOP: the category of topological spaces and continuous functions.

TOP_i: the category of topological spaces satisfying the T_i axiom $i = 0, 1, 2$.

Ury: the category of Urysohn spaces (points are separated by disjoint closed neighborhoods).

TOP_3: the category of regular Hausdorff spaces.

Tych: the category of completely regular Hausdorff spaces.

0-dym: the category of zero-dimensional spaces (i.e. Hausdorff spaces with a base of clopen sets).

All subcategories listed in **0.2** are epireflective subcategories of TOP.

The following results can be found in $[4]_{1,2}$.

(1) $\tau_{\mathcal{A}} \leq \tau$ for all $(X, \tau) \in \mathcal{A}$ iff $\mathcal{A} \subset \text{TOP}_2$.

(2) For $\mathcal{A} = \text{TOP}_2$, TOP_3, Tych, 0-dym, $\tau_{\mathcal{A}} = \tau$ for each $(X, \tau) \in \mathcal{A}$.

(3) The TOP_0-closure is the front-closure defined on **[11]** $\text{Frcl}(A) = \{x \in X$: for each open nhood U of x, $\overline{\{x\}} \cap U \cap A \neq \emptyset\}$.

(4) The TOP_1-closure is the identity for all T_1-spaces.

(5) For $\mathcal{A} = \text{Ury}$ let $X \in \mathcal{A}$ and $M \subset X$, we define $\text{cl}_\theta(M) = \{x \in X$: for each nhood V of x, $\overline{V} \cap M \neq \emptyset\}$, this is the θ-closure introduced by Velichko [14]. For $X \in \text{Ury}$ and $M \subset X$ we have $\text{cl}_\theta M \subset [M]^X_{\text{Ury}}$ and $M = \text{cl}_\theta(M)$ iff $M = [M]^X_{\text{Ury}}$, thus the Ury-closure is the idempotent hull of cl_θ.

1 - $\mathcal{A}$-Lindelof space

Def. 1.1. Let $\mathcal{A}$ be an epireflective subcategory of TOP. $(X, \tau) \in \mathcal{A}$ is said to be *$\mathcal{A}$-Lindelof* if $(X, \tau_{\mathcal{A}})$ is a Lindelof space.

Def. 1.2. [5]$_2$ Let $\mathcal{A}$ be an epireflective subcategory of TOP. $(X, \tau) \in \mathcal{A}$ is said to be *$\mathcal{A}$-compact* if $(X, \tau_{\mathcal{A}})$ is compact.

The following classes are denoted as follows:

$\mathcal{A}$Lind: the class of Lindelof spaces X such that $X \in \mathcal{A}$.

$L_{\mathcal{A}}$: the class of $\mathcal{A}$-Lindelof spaces.

$K_{\mathcal{A}}$: the class of $\mathcal{A}$-compact spaces.

Obviously $K_{\mathcal{A}} \subset L_{\mathcal{A}}$ for each $A \subset$ TOP.

Let Ind and Discr be the categories of indiscrete spaces and discrete spaces respectively, and let Singol be the category of topological spaces whose underlying set has at most one element.

Theorem 1.3. *Let A be a non empty epireflective subcategory of* TOP. *Then the following conditions are equivalent:*

(a) $K_{\mathcal{A}} \subsetneqq L_{\mathcal{A}}$. (b) $\mathcal{A} \neq$ Singol *and* $\mathcal{A} \neq$ Ind. (c) 0-dim $\subset \mathcal{A}$.

Proof. Obviously (a) $\Rightarrow$ (b), (b) $\Rightarrow$ (c) follows from the fact that 0-dim is the smallest epireflective subcategory of TOP different from Singol.

Now let $(X, \tau) \in$ 0-dim $\subset \mathcal{A}$ hence the $\mathcal{A}$-closure is finer than the 0-dim closure in (X, τ), but it is well known that the 0-dim-closure is the ordinary closure in (X, τ), therefore $\tau_{\mathcal{A}} \geqslant \tau$. If (X, τ) is a countably infinite discrete space we have $\tau_{\mathcal{A}} = \tau$, hence (X, τ) is $\mathcal{A}$-Lindelof but it is not $\mathcal{A}$-compact, therefore $K_{\mathcal{A}} \subsetneqq L_{\mathcal{A}}$.

Remarks 1.4. (a) For each $(X, \tau) \in \text{TOP}_0$, (X, τ_{TOP_0}) is a T_3-space [4]$_2$, hence if (X, τ) is TOP_0-Lindelof then (X, τ_{TOP_0}) is paracompact.

(b) If $(X, \tau) \in L_{\text{Ury}}$ and $(X, \tau_{\text{Ury}}) \in \text{TOP}_3$ then (X, τ) is a functionally Hausdorff space (i.e. points are separated by continuous real valued maps). In fact we have that (X, τ_{Ury}) is a Lindelof T_3-space hence it is a T_4-space, therefore (X, τ_{Ury}) is functionally Hausdorff, but $\tau_{\text{Ury}} \leqslant \tau$ and this property is closed under refinements hence (X, τ) is a functionally Hausdorff space.

Let Haus$(\{X_j\}) = \{X \in \text{TOP}$ such that every continuous mapping $f: X_j \to X$ is constant$\}$, where X_j is a T_1-space with cofinite topology and infinite cardinality, [9], and let $LM - T_2$ be the category of Lawson-Madison spaces (a topological space X is $LM - T_2$ iff every compact subspace of X is T_2, [10], [9]).

For $\mathcal{A} = \text{TOP}_0$, TOP_1, Haus $(\{X_j\})$, $LM - T_2$ if $(X, \tau) \in \mathcal{A}$ then $\tau \leq \tau_{\mathcal{A}}$ hence we have $L_{\mathcal{A}} \subset \mathcal{A}$Lind.

For $\mathcal{A} \subset \text{TOP}_2$ we have $\tau_{\mathcal{A}} \leq \tau$ for each $(X, \tau) \in \mathcal{A}$ hence we have $\mathcal{A}\text{Lind} \subset L_{\mathcal{A}}$.

Examples 1.5. (a) If (X, τ) is a T_D-space [2] (every point is the intersection of a closed and an open set) then (X, τ_{TOP_0}) is a discrete space $[4]_2$, hence every uncountable Lindelof T_D-space is not TOP_0-Lindelof, therefore $L_{\text{TOP}_0} \subsetneqq \text{TOP}_0\,\text{Lind}$.

(b) For $\mathcal{A} = \text{TOP}_1$, Haus$(\{X_j\})$ if $(X, \tau) \in \mathcal{A}$ then $(X, \tau_{\mathcal{A}})$ is discrete $[4]_{1,3}$, hence every uncountable Lindelof space $(X, \tau) \in \mathcal{A}$ is not $\mathcal{A}$-Lindelof, therefore $L_{\mathcal{A}} \subsetneqq \mathcal{A}$Lind.

(c) $L_{\mathcal{A}} = \mathcal{A}$Lind for $\mathcal{A} = \text{TOP}_2$, TOP_3, Tych, 0-dim.

(d) There exists a Ury-compact (hence Ury-Lindelof) space (X, τ) such that it is countably compact but it is not compact ($[4]_4$, Ex. 5), hence (X, τ) is not a Lindelof space, therefore UryLind $\subsetneqq L_{\text{Ury}}$.

Theorem 1.6. *The class L_{LM-T_2} is strictly smaller than the class LM-T_2 Lind.*

Proof. Let (X, τ) be an uncountable space with the co-countable topology (i.e. a proper subset is closed iff it is countable), (X, τ) is a $LM - T_2$ space (since every compact subset is finite, [9]).

Obviously for each compact space P and for every continuous map $f: P \to (X, \tau)$ $f(P)$ is a closed discrete subspace of (X, τ) hence by Prop. 1.11 in [6] it follows that (X, τ_{LM-T_2}) is discrete, hence $\tau \subsetneqq \tau_{LM-T_2}$.

Now let $S = \{S_i\}_{i \in I}$ be a τ-open cover of X, if $S_{i_0} \in S$ then $X - S_{i_0} = \{x_j\}_{j=1}^{\infty}$ (since it is closed); for each $x_j \in X - S_{i_0}$ let S_j be an open set in S such that $x_j \in S_j$, then $S_{i_0} \cup \{S_j\}_{j=1}^{\infty}$ *is a countable subcover of* (X, τ), hence (X, τ) is a Lindelof space. But (X, τ_{LM-T_2}) is an uncountable discrete space hence it is not a Lindelof space, i.e. $(X, \tau) \in LM - T_2\,\text{Lind}$ but $(X, \tau) \notin L_{LM-T_2}$.

Remarks 1.7. (a) A space X is $\mathcal{A}$-Lindelof iff every family of $\mathcal{A}$-closed subsets of X with the countable intersection property has a non empty intersection.

(b) Let $(X, \tau) \in L_{\mathcal{A}}$ and $(Y, \sigma) \in \mathcal{A}$. If $f:(X, \tau) \to (Y, \sigma)$ is $\mathcal{A}$-continuous and onto then (Y, σ) is $\mathcal{A}$-Lindelof.

We will denote by $T(\mathcal{A})$ the class of topological spaces $(X, \tau) \in \mathcal{A}$ such that every compact subset is $\mathcal{A}$-closed in (X, τ). For $\mathcal{A} = \text{TOP}_1$, Haus$(\{X_j\})$, $LM - T_2$, TOP_2, Ury, TOP_3, Tych, 0-dim we have $T(\mathcal{A}) = \mathcal{A}$, while $T(\text{TOP}_0)$ is strictly contained between TOP_1 and TOP_0 [5]$_1$.

Proposition 1.8. (1) *Let (X, τ) be an $\mathcal{A}$-Lindelof space such that $(X, \tau) \in T(\mathcal{A})$. Then for each uncountable subset S of X there exists $x \in X$ such that $S - \{x\}$ is not compact.*

(2) *Let (X, τ) be a TOP_0-Lindelof space such that $(X, \tau) \in T(\text{TOP}_0)$. If (X, τ) is compact then X is countable.*

Proof. (1) Let S be an uncountable subset of X, since $(X, \tau_{\mathcal{A}})$ is a Lindelof space then there exists a $\tau_{\mathcal{A}}$-accumulation point of S (16.D.2, [16]), i.e. there exists $x \in X$ such that $x \in cl_{\tau_{\mathcal{A}}}(S - \{x\})$, where $cl_{\tau_{\mathcal{A}}}$ is the ordinary closure in $(X, \tau_{\mathcal{A}})$.

If $S - \{x\}$ is compact then it is $\mathcal{A}$-closed in (X, τ) (because $(X, \tau) \in T(\mathcal{A})$), hence $x \in cl_{\tau_{\mathcal{A}}}(S - \{x\}) = S - \{x\}$, a contradiction.

(2) Let X be uncountable. From (1) we have that there exists $x \in X$ such that $X - \{x\}$ is not TOP_0-closed.

Since $(X, \tau) \in T(\text{TOP}_0)$ then $X - \{x\}$ is not compact. Let us suppose that (X, τ) is compact.

Let $(U_\alpha)_{\alpha \in A}$ be an open cover of $X - \{x\}$ which has no finite subcover.

Obviously $x \notin \bigcup_{\alpha \in A} U_\alpha$ (otherwise we have a finite subcollection of $(U_\alpha)_{\alpha \in A}$ covering X and, a fortiori, $X - \{x\}$), hence $X - \{x\} = \bigcup_{\alpha \in A} U_\alpha$.

Then $X - \{x\}$ is τ-open, but every τ-open set is TOP_0-closed, a contradiction.

A topological space $(X, \tau) \in \mathcal{A}$ is called *$\mathcal{A}$-minimal* if $\tau' \leq \tau$ and $(X, \tau') \in \mathcal{A}$ imply $\tau' = \tau$.

Let us denote by TOP_4 the class of T_4-spaces.

Proposition 1.9. *Let $\text{TOP}_4 \subset \mathcal{A} \subset \text{TOP}_2$. Let (X, τ) be $\mathcal{A}$-minimal and $(X, \tau_{\mathcal{A}}) \in T_3$. (X, τ) is $\mathcal{A}$-Lindelof if and only if it is a Lindelof space.*

Proof. Let (X, τ) be a Lindelof space, since $\mathscr{A} \subset \mathrm{TOP}_2$ we have that $\tau_{\mathscr{A}} \leq \tau$, therefore (X, τ) is $\mathscr{A}$-Lindelof. If (X, τ) is an $\mathscr{A}$-Lindelof space then $(X, \tau_{\mathscr{A}})$ is T_3 and Lindelof, hence $(X, \tau_{\mathscr{A}}) \in \mathrm{TOP}_4$. Since $\tau_{\mathscr{A}} \leq \tau$ and (X, τ) is $\mathscr{A}$-minimal we have that $\tau = \tau_{\mathscr{A}}$, therefore (X, τ) is a Lindelof space.

Remark 1.10. We recall that for each $(X, \tau) \in \mathrm{TOP}_0$, $\tau_{\mathrm{TOP}_0} = \sup(\tau, \tau_-)$ where τ_- is the topology on X such that every x in X has $cl_\tau x$ (where cl_τ is the ordinary closure in (X, τ)) as its smallest τ_--open neighbourhood.

The following problem arises: let $(X, \tau) \in \mathrm{TOP}_0$, is $(X, \tau_{\mathrm{TOP}_0})$ a Lindelof space if and only if (X, τ), (X, τ_-) are Lindelof?

One implication is obvious, in fact let $(X, \tau_{\mathrm{TOP}_0})$ be a Lindelof space then from the continuity of the identities i_1: $(X, \tau_{\mathrm{TOP}_0} \to (X, \tau)$ and i_2: $(X, \tau_{\mathrm{TOP}_0}) \to (X, \tau_-)$ follows that (X, τ) and (X, τ_-) are Lindelof. The converse is not true: let X be the set of real numbers and let the closed sets be (besides $\emptyset$ and X) all $\{x\}$ for $x \neq 0$ and all finite unions of these sets and $\cup \{x : x \neq 0\}$, this space (X, τ) is compact (hence it is Lindelof) and T_D. (X, τ_-) is compact (hence it is Lindelof), in fact is the only open set containing $x = 0$, but $(X, \tau_{\mathrm{TOP}_0})$ is an uncountable discrete space (because (X, τ) is T_D) hence it is not Lindelof.

2 - $\mathscr{A}$-countably compact spaces

Def. 2.1. Let $\mathscr{A}$ be an epireflective subcategory of TOP. $(X, \tau) \in \mathscr{A}$ is said to be *$\mathscr{A}$-countably compact* if $(X, \tau_{\mathscr{A}})$ is countably compact.

We will denote by $C_{\mathscr{A}}$ the class of $\mathscr{A}$-countably compact spaces and by $\mathscr{A}$countcomp the class of countably compact spaces X such that $X \in \mathscr{A}$.

Obviously a space is $\mathscr{A}$-compact if and only if it is $\mathscr{A}$-Lindelof and $\mathscr{A}$-countably compact.

For $\mathscr{A} = \mathrm{TOP}_0$, TOP_1, Haus $(\{X_j\})$, $LM\text{-}T_2$ if $(X, \tau) \in \mathscr{A}$ then $\tau \leq \tau_{\mathscr{A}}$ hence we have $C_{\mathscr{A}} \subset \mathscr{A}$CountComp.

Examples 2.2. (a) If (X, τ) is an infinite countably compact T_D-space then $(X, \tau_{\mathrm{TOP}_0})$ is an infinite discrete space hence $(X, \tau) \notin C_{\mathrm{TOP}_0}$, therefore $C_{\mathrm{TOP}_0} \subsetneqq \mathrm{TOP}_0$ CountComp.

(b) For $\mathscr{A} = \mathrm{TOP}_1$, Haus $(\{X_j\})$ if $(X, \tau) \in \mathscr{A}$ then $(X, \tau_{\mathscr{A}})$ is discrete, hence every infinite countably compact space $(X, \tau) \in \mathscr{A}$ is not $\mathscr{A}$-countably compact, therefore $C_{\mathscr{A}} \subsetneqq \mathscr{A}$CountComp.

Proposition 2.3. *The class C_{LM-T_2} is strictly smaller than the class* $LM-T_2$CountComp.

Proof. There exists a countably compact subspace (X, τ) of βN (hence (X, τ) belongs to $LM-T_2$CountComp), described by Walker ([15], p. 189), such that it is uncountable and every compact subset is finite [12].

Obviously for each compact space P and for every continuous mapping $f:P\to(X, \tau)$ $f(P)$ is a closed discrete subspace of (X, τ) hence by Prop. 1.11 in $[4]_3$ it follows that (X, τ_{LM-T_2}) is an infinite discrete space, therefore (X, τ_{LM-T_2}) is not countably compact, i.e. $(X, \tau)\notin C_{LM-T_2}$.

For $\mathscr{A}=\mathrm{TOP}_2$, TOP_3, Tych, 0-dim we have $C_{\mathscr{A}}=\mathscr{A}$CountComp.

For each $(X, \tau)\in$ Ury we have $\tau_{\mathrm{Ury}}\leqslant\tau$ hence UryCountComp $\subset C_{\mathrm{Ury}}$, moreover the space described in $[4]_4$ (Example 2) is Ury-countably compact but it is not countably compact therefore UryCountComp $\subsetneqq C_{\mathrm{Ury}}$. For each $\mathscr{A}\subset\mathrm{TOP}$ we have $K_{\mathscr{A}}\subset C_{\mathscr{A}}$.

Example 2.4. (a) For $\mathscr{A}=\mathrm{TOP}_1$, Haus $(\{X_j\})$ we have $K_{\mathscr{A}}=C_{\mathscr{A}}$.

(b) Let $\mathscr{A}=LM-T_2$, the space $(X, \tau)=\Omega-\{\omega_1\}$, i.e. the space of countable ordinals, is T_2 locally compact (hence $(X, \tau)\in\mathscr{A}$ and it is a k-space, therefore $\tau=\tau_{\mathscr{A}}$, [6]) and moreover it is a non compact countably compact space (17.2.c [16]), hence $(X, \tau)\in C_{\mathscr{A}}$ but it is not an $\mathscr{A}$-compact space.

(c) The space (X, τ) considered in (b) is a T_3-space hence $\tau=\tau_{\mathrm{Ury}}$, $([4]_1)$, therefore $(X, \tau)\in C_{\mathrm{Ury}}$ but it is not Ury-compact.

We don't know if there exists a T_0-space (X, τ) such that $(X, \tau_{\mathrm{TOP}_0})$ is a non compact countably compact space.

The space (X, τ) described in 1.10 is an example of a countably compact space such that (X, τ_-) is countably compact and $(X, \tau_{\mathrm{TOP}_0})$ is an uncountable discrete space.

Remarks 2.5. (a) $X\in C_{\mathscr{A}}$ iff every countable family of $\mathscr{A}$-closed subsets of X with the finite intersection property has a non empty intersection.

(b) Let $(X, \tau)\in C_{\mathscr{A}}$ and $(Y, \sigma)\in\mathscr{A}$. If $f:(X, \tau)\to(Y, \sigma)$ is $\mathscr{A}$-continuous and onto then $(Y, \sigma)\in C_{\mathscr{A}}$.

Proposition 2.6. (1) *Let (X, τ) be an $\mathscr{A}$-countably compact space such that $(X, \tau)\in T(\mathscr{A})$, then for each infinite subset S of X there exists $x\in X$ such $S-\{x\}$ is not compact.*

(2) *Let* (X, τ) *be a* TOP_0*-countably compact space such that* $(X, \tau) \in T(\text{TOP}_0)$. (X, τ) *is compact if and only if* X *is finite.*

Proof. 1. Let S be an infinite subset of X. Since $(X, \tau_{\mathcal{A}})$ is a T_1 countably compact space then there exists a $\tau_{\mathcal{A}}$-accumulation point of S (17.F.2., [16]). i.e. there exists $x \in X$ such that $x \in cl_{\tau_{\mathcal{A}}}(S - \{x\})$, where $cl_{\tau_{\mathcal{A}}}$ is the ordinary closure in $(X, \tau_{\mathcal{A}})$. If $S - \{x\}$ is compact then it is $\mathcal{A}$-closed in (X, τ) (because $(X, \tau) \in T(\mathcal{A})$), hence $x \in cl_{\tau_{\mathcal{A}}}(S - \{x\}) = S - \{x\}$, a contradiction.

(2) The sufficiency is obvious. The proof of the necessity is the same of 1.8.2.

Proposition 2.7. *Let* $\text{TOP}_3 \subset \mathcal{A} \subset \text{TOP}_2$. *Let* (X, τ) *be an* $\mathcal{A}$*-minimal space such that* $(X, \tau_{\mathcal{A}})$ *is* T_2 *and first countable.* (X, τ) *is* $\mathcal{A}$*-countably compact if and only if it is countably compact.*

Proof. Let (X, τ) be a countably compact space, since $\mathcal{A} \subset \text{TOP}_2$ we have that $\tau_{\mathcal{A}} \leq \tau$, therefore (X, τ) is $\mathcal{A}$-countably compact. If (X, τ) is $\mathcal{A}$-countably compact then $(X, \tau_{\mathcal{A}})$ is a T_2 countably compact first countable space, hence it is a T_3-space [1], therefore $(X, \tau_{\mathcal{A}}) \in \mathcal{A}$.

Since $\tau_{\mathcal{A}} \leq \tau$ and (X, τ) is $\mathcal{A}$-minimal we have that $\tau_{\mathcal{A}} = \tau$, hence (X, τ) is countably compact.

In [7] Hanai proved that a Hausdorff space X is countably compact iff the projection $p: X \times N^+ \to N^+$ is closed, where N^+ is the Alexandroff one-point compactification of the discrete space of the natural numbers N.

Now we prove the following

Theorem 2.8. *Let* 0-dim $\subset \mathcal{A} \subset \text{TOP}_2$. *If* $F_{\mathcal{A}}$ *is finitely multiplicative then the following conditions are equivalent*: (a) (X, τ) *is* $\mathcal{A}$*-countably compact;* (b) *the projection* $p: (X, \tau) \times N^+ \to N^+$ *is* $\mathcal{A}$*-closed.*

Proof. First we prove that $N^+ = F_{\mathcal{A}}(N^+)$. In fact if $(Y, \sigma) = N^+ \in$ 0-dim $\subset \mathcal{A}$ then by $\mathcal{A} \subset \text{TOP}_2$ it follows that $\sigma_{\mathcal{A}} \leq \sigma$, but $\sigma_{\mathcal{A}}$ is a Hausdorff topology (because $F_{\mathcal{A}}$ is finitely multiplicative $[4]_2$) and σ is a compact Hausdorff topology hence it is TOP_2-minimal therefore $\sigma_{\mathcal{A}} = \sigma$, i.e. $N^+ = F_{\mathcal{A}}(N^+)$.

Now let (X, τ) be an $\mathcal{A}$-countably compact space, then $(X, \tau_{\mathcal{A}})$ is a Hausdorff countably compact space, hence by Hanai theorem we have that the projection $p = F_{\mathcal{A}}(p): (X, \tau_{\mathcal{A}}) \times N^+ \to N^+$ is closed, since $(X, \tau_{\mathcal{A}}) \times N^+ = F_{\mathcal{A}}[(X, \tau) \times N^+]$ then the projection $p: (X, \tau) \times N^+ \to N^+$ is $\mathcal{A}$-closed.

Conversely if the projection $p:(X, \tau)\times N^+\to N^+$ is $\mathcal{A}$-closed then $p=F_{\mathcal{A}}(p):(X, \tau_{\mathcal{A}})\times N^+\to N^+$ is closed, hence by Hanai theorem we have that $(X, \tau_{\mathcal{A}})$ is countably compact, i.e. (X, τ) is $\mathcal{A}$-countably compact.

Remark. F_{TOP_0} is finitely multiplicative but the Theorem 2.8 is not true for $\mathcal{A}=\mathrm{TOP}_0$.

In fact the projection $p:(X, \tau)\times N^+\to N^+$ is always TOP_0-closed because N^+ is a T_D-space).

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Sommario

Per ogni classe $\mathcal{A}$ di spazi topologici esiste un operatore di chiusura $[\,]_{\mathcal{A}}: P(X) \to P(X)$, detto $\mathcal{A}$-chiusura, dove X è uno spazio topologico e $P(X)$ è l'insieme potenza di X. In un precedente lavoro abbiamo introdotto il concetto di compattezza relativa ad una classe $\mathcal{A}$ di spazi topologici (in breve $\mathcal{A}$-compattezza) ed abbiamo mostrato che gli spazi $\mathcal{A}$-compatti (cioè gli spazi $(X, \tau) \in \mathcal{A}$ tali che la topologia $\tau\mathcal{A}$ su X generata dalla $\mathcal{A}$-chiusura è compatta) hanno un ruolo significativo in $\mathcal{A}$. Lo scopo del presente lavoro è di studiare gli spazi $(X, \tau) \in \mathcal{A}$ tali che la topologia $\tau\mathcal{A}$ su X è di Lindelöf oppure numerabilmente compatta.

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