Riv. Mat. Univ. Parma (4) 15 (1989), 105-110

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Compact almost contact solvmanifolds admitting neither Sasakian nor cosymplectic structures (**)

1 - Introduction

Let M be an odd dimensional compact manifold. As it is well known, there are topological obstructions to the existence on M of either a Sasakian or a cosymplectic structure. In fact, if M has a Sasakian structure then its Betti numbers verify some conditions [3]. On the other hand M has a cosymplectic structure if and only if $M \times S^1$ has a canonically associated Kähler structure and the odd Betty numbers of $M \times S^1$ must be even.

Let H be the 3-dimensional Heisenberg group H. As it is well known, the quotient compact nilmanifold $M = \Gamma \backslash H$ (where Γ is a discrete subgroup of H) has a Sasakian structure, but it does not admit a cosymplectic structure since $M \times S^1$ has no Kähler structure (see [5]).

In this paper, we consider a compact solvmanifold $M(k) = D_1/S_1$, where S_1 is a 3-dimensional solvable non-nilpotent Lie group and D_1 a discrete subgroup of S_1 (see [1]). We prove the following nonexistence results:

(1) $M(k)$ can have no Sasakian structures

(2) $M(k)$ can have no cosymplectic structures

(3) $M(k)$ can have no regular contact structures.

Finally, we construct two examples of almost contact metric structures on $M(k)$ (for instance, we define a quasi-Sasakian structure on $M(k)$).

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(**) Ricevuto: 28-II-1989.

2 - Almost contact metric manifolds

Let $(M, \phi, \eta, \xi, \langle,\rangle)$ be an almost contact metric manifold of dimension $2n+1$. The fundamental 2-form Φ of M is defined by

$$\Phi(U, V) = \langle U, \phi V\rangle \qquad U, V \in \chi(M).$$

Let us recall some well known definitions. The almost contact metric manifold M is said to be:

contact iff $\Phi = d\eta$;

normal iff $[\phi, \phi] + d\eta \otimes \xi = 0$;

Sasakian iff it is contact and normal;

almost cosymplectic iff $d\Phi = d\eta = 0$;

cosymplectic iff it is normal and almost cosymplectic;

quasi-Sasakian iff it is normal and $d\Phi = 0$;

semi-cosymplectic iff $\delta\Phi = \eta = 0$.

If M is an almost contact metric structure, then $M \times S^1$ is an almost Hermitian manifold. As it is well known, M is normal (resp. cosymplectic) iff $M \times S^1$ is complex (resp. Kähler). Moreover, we have the following result concerning the topology of Sasakian manifolds:

Theorem 1. [3] Let M be a compact Sasakian manifold of dimension $2n+1$. Then the p-th Betti number b_p of M is even if p is odd and $p \leqslant n$, and b_p is even if p is even for $p \leqslant n+1$.

Finally, let M be a contact manifold with contact form η. M is said to be *regular* iff the *characteristic vector field* ξ of M is regular, that is every point $x \in M$ has a cubical coordinate neighborhood U such that the integral curves of ξ passing through U pass through the neighborhood only once.

We have the following theorem of Boothy and Wang [4]:

Theorem 2. Let M be a compact regular contact manifold M of dimension $2n+1$. Then M is a principal circle bundle over a $2n$-dimensional symplectic manifold N ($M \to N$ is the *Boothby-Wang fibration* of M).

3 - The manifolds $M(k)$

Let S_1 be the 3-dimensional solvable non-nilpotent Lie group of matrices of the form

$$\begin{pmatrix} e^{kz} & 0 & 0 & x \\ 0 & e^{-kz} & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $x,\ y,\ z \in R$ and k is a real number such that $e^k + e^{-k}$ is an integer number different to 2. A standard computation shows that

$$\{dx - kz\,dz, \quad dy + ky\,dz, \quad dz\}$$

is a basis for the right invariant 1-forms on S_1. Now, let D_1 be a discrete subgroup of S_1 such that the quotient space $M(k) = D_1 \diagdown S_1$ is compact (see [1]). Then there exists a global basis of 1-forms $\{\alpha,\ \beta,\ \gamma\}$ on $M(k)$ such that

$$\pi^* \alpha = dx - kz\,dz \qquad \pi^* \beta = dy + kz\,dz \qquad \pi^* \gamma = dz$$

where $\pi : S_1 \to M(k)$ is the canonical projection. Then we have

$$\text{(1)} \qquad d\alpha = -k\alpha \wedge \gamma \qquad d\beta = k\beta \wedge \gamma \qquad d\gamma = 0\,.$$

Therefore, if $\{X,\ Y,\ Z\}$ is the dual basis of vector fields to $\{\alpha,\ \beta,\ \gamma\}$ we have

$$\text{(2)} \qquad [X,\ Y] = 0 \quad [X,\ Z] = kX \quad [Y,\ Z] = -kY\,.$$

From (1) we easily prove the following

Proposition 1. *The Betti numbers of $M(k)$ are*

$$b_0(M(k)) = b_1(M(k)) = b_2(M(k)) = b_3(M(k)) = 1\,.$$

Then, from Proposition 1, we have

Theorem 3. *$M(k)$ can have no Sasakian structures.*

Furthermore, the product manifold $M(k)\times S^1$ can have no complex (and hence, no Kähler) structures. The key for this is Yau's theorem [10] (see [7], [6]). Thus, we deduce

Theorem 4. *$M(k)$ can have no cosymplectic structures.*

Remark 1. It is clear that $M(k)$ can have no normal structures since $M(k)\times S^1$ can have no complex structures.

In [8], Martinet proved that every compact orientable 3-dimensional manifold carries a contact structures. Here we will just give explicitly a contact structure on $M(k)$. Define $\eta=\alpha+\beta$; then $\eta\wedge d\eta\neq 0$. Thus, η is a contact form on $M(k)$. However, we have

Theorem 5. *$M(k)$ can have no regular contact structures.*

Proof. If $M(k)$ admitted a regular contact structure, $M(k)$ would be a principal circle bundle over a symplectic 2-dimensional manifold N by Theorem 2. In this case, the first Betti numbers of $M(k)$ and of N are equal (see [9]). However, since N is symplectic (and hence orientable) then N is S^2 or a g-torus T_g. Therefore, $b_1(N)=0$ or $2g$, a contradiction.

Remark 2. We notice that the same is true for the 3-dimensional torus T^3 (see [2]). Moreover, $M(k)$ can be seen as the bundle space of a 2-torus over the circle S^1 (see [6]).

4 - Examples of almost contact metrics structures on $M(k)$

In spite of the nonexistence theorems proved in 3, there exist interesting almost contact metric structures on $M(k)$. In this section we construct some of those structures.

1. Define

$$\phi=\alpha\otimes Y-\beta\otimes X \qquad \xi=Z \qquad \eta=\gamma \qquad \langle,\rangle=\alpha^2+\beta^2+\gamma^2 .$$

Then $(\phi,\ \xi,\ \eta,\ \langle,\rangle)$ is an almost contact metric structure on $M(k)$. Its fundamental 2-form is given by $\Phi = -\alpha \wedge \beta$. Hence $d\Phi = d\eta = 0$. Thus $(\phi, \xi, \eta, \langle,\rangle)$ is a *non-normal almost cosymplectic structure on M.*

2. Define

$$\phi = \alpha \otimes Z - \gamma \otimes X \qquad \xi = Y \qquad \eta = \beta \qquad \langle,\rangle = \alpha^2 + \beta^2 + \gamma^2 .$$

Then $(\phi, \xi, \eta, \langle,\rangle)$ is an almost contact metric structure on $M(k)$ whose fundamental 2-form is given by $\Phi = -\alpha \wedge \gamma$. A simple computation from (1) and (2) shows that

$$[\phi, \phi] = 0 \qquad d\Phi = 0 \qquad \delta\eta = 0 \qquad \delta\Phi = -k\alpha$$

$$d\eta = k\beta \wedge \gamma \qquad (d\eta \otimes \xi)(Y, Z) = kY .$$

Therefore $(\phi, \xi, \eta, \langle,\rangle)$ is *neither normal nor semi-cosymplectic structure.*

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Sommario

Si costruisce una famiglia $M(k)$ di «solvmanifolds» di dimensione 3 e si dimostra che $M(k)$ non ammette nè strutture sasakiane nè strutture cosimplettiche.

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